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# HIGH CONTRASTING DIFFUSION IN HEISENBERG GROUP: HOMOGENIZATION OF OPTIMAL CONTROL VIA UNFOLDING*

A. K. NANDAKUMARAN† AND ABU SUFIAN‡

**Abstract.** The periodic unfolding method is one of the latest tools for studying multiscale problems like homogenization after the development of multiscale convergence in the 1990s. It provides a good understanding of various microscales involved in the problem, which can be conveniently and easily applied to get the asymptotic limit. In this article, we develop *the periodic unfolding* for the Heisenberg group, which has a noncommutative group structure. The concept of greatest integer part and fractional part for the Heisenberg group has been introduced corresponding to the periodic cell. Analogous to the Euclidean unfolding operator, we prove the integral equality, $L^2$-weak compactness, unfolding gradient convergence, and other related properties. Moreover, we have the adjoint operator for the unfolding operator, which can be recognized as an average operator. As an application of the unfolding operator, we have homogenized the standard elliptic PDE with oscillating coefficients. We have also considered an optimal control problem with the state equation having high contrasting diffusivity coefficients. The high contrasting coefficients are an added difficulty in the analysis. Moreover, we have characterized the interior periodic optimal control in terms of the unfolding operator, which helps us to analyze the asymptotic behavior.

**Key words.** homogenization, periodic unfolding, two-scale convergence

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## 1. Introduction.

The mathematical theory of homogenization was introduced in the 1970s to describe the behavior of composite materials. Since then, several homogenization methods have been developed for the Euclidean setting, which has commutative group structures. The research on homogenization in noncommutative group structures is very limited. Among the early results on the homogenization in noncommutative group structures, we cite the results by Biroli, Mosco, and Tchou [9]. In this paper, the authors construct explicitly a periodic tilling associated with the Laplace operator $\Delta_{\mathbb{H}}$ on the Heisenberg group. They have analyzed the asymptotic behavior of its eigenfunctions in a domain with isolated Heisenberg periodic holes with Dirichlet boundary conditions on their boundaries. To establish the convergence to the homogenized problem, they employ Tartars energy method. Another piece of work on homogenization in the Heisenberg group is due to Biroli, Tchou, and Zhikov in [10]. The problem has been revisited in [13] with less regular holes. Due to less regularity on holes, they could not employ the method as in [10], and they used the method introduced in [29] by Zhikov.

The Γ-convergence is another useful tool in homogenization theory. The Γ-convergence is well known for functionals that rely on Euclidean vector fields but not for general vector fields. Maione, Pinamonti, and Cassano demonstrated the

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†Indian Institute of Science, Bangalore, India (nands@iisc.ac.in).
‡TIFR-CAM, Bangalore, Karnataka, 560065, ID (abu22@tifrbng.res.in).

Γ-compactness results for nonlinear functionals depending on general vector fields, which in particular include Heisenberg vector fields, in [20]. They later investigated a periodic homogenization problem in [19] and proved a Γ-compactness theorem for linear second order differential operators depending on general vector fields in Carnot groups. For further reading in this direction, we refer to the articles [5, 12, 22] and references therein.

Spanglo's $G$-convergence and its extension called $H$-convergence by F. Murat and L. Tartar are very important tools for multiscale analysis of differential operators with nonperiodic coefficients in the Euclidean setup. The main tools to prove $H$-compactness and $G$-compactness are the div-curl lemma and the compensated compactness. In Carnot groups, Baldi, Franchi, and Tesi in [7, 8] have proved the div-curl lemma and compensated compactness and eventually proved the $G$-compactness and $H$ compactness. Further, the notion of $H$-convergence and $G$-convergence extended by Maione, Paronetto, and Vecchi for more general differential operator in Carnot group in [18, 21]. For further reading, we refer to the articles [14, 15] and references therein.

Now, coming back to the Euclidean setting, among many methods developed in the last 50 years, two-scale convergence and unfolding methods are very effective techniques. The two-scale convergence was introduced by Nguetseng [27] and later developed by Allaire in [4] which has been extensively applied by various authors over the last few decades. The method of two-scale convergence in $\mathbb{R}^n$ is deeply related to the group structure of $\mathbb{R}^n$ and the definition of the periodic functions in terms of group translation.

The concept of the tiling and the periodic functions defined in [9] for the Heisenberg group motivated Franchi and Tesi in [13] to define the concept of two-scale convergence in the Heisenberg group. As an application of this two-scale convergence, they have investigated a Dirichlet problem for a generalized Kohn Laplacian operator with highly oscillating Heisenberg-periodic coefficients in a domain perforated by interconnected Heisenberg-periodic pipes. They have proved all the similar results as in Euclidean two-scale convergence.

One of the latest methods for homogenization is *the periodic unfolding method* introduced by Cioranescu, Damlamian, and Griso in [16], where the microscale is introduced at the microlevel of the problem before taking the limit, whereas, in two-scale convergence, the microscale is recovered at the limit. The unfolding operator is also quite easy to apply in multiscale analysis and helps to see more deeply the microscopic scale. For the reader's sake, we recall the two-scale convergence and unfolding operators in the Euclidean space setup and will see how they are related to each other.

Let $Y$ be the reference cell $[0,1)^n$ and $\Omega$ be a bounded domain of $\mathbb{R}^n$. The smooth $Y$ periodic (Euclidean sense) function space is denoted by $C^\infty_\#(Y)$.

Definition 1 (two-scale convergence). *A family of functions* $\{u_\varepsilon\} \subset L^2(\Omega)$ *is said to be two-scale convergent to* $u_0 \in L^2(\Omega \times Y)$ *if for any* $\psi \in C^\infty_c(\Omega; C^\infty_\#(Y))$*, we have*

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\psi\left(x,\frac{x}{\varepsilon}\right)dx = \frac{1}{|Y|}\int_{\Omega\times Y} u_0(x,y)\psi(x,y)\,dxdy. \tag{1.1}$$

Let $E_\varepsilon = \{k \in \mathbb{Z}^n : \varepsilon k + \varepsilon Y \subset \Omega\}$, $\Omega_\varepsilon = \bigcup_{k\in E_\varepsilon}\{k\varepsilon + \varepsilon Y\}$, $\Lambda_\varepsilon = \Omega\backslash\Omega_\varepsilon$ (Figure 1). The greatest integer part and fractional part with respect to $Y$ are denoted by $\left[\frac{x}{\varepsilon}\right]_Y$ and $\left\{\frac{x}{\varepsilon}\right\}_Y$, respectively. Note that the microscale $y$ is given in the limit $u_0 = u_0(x,y)$,

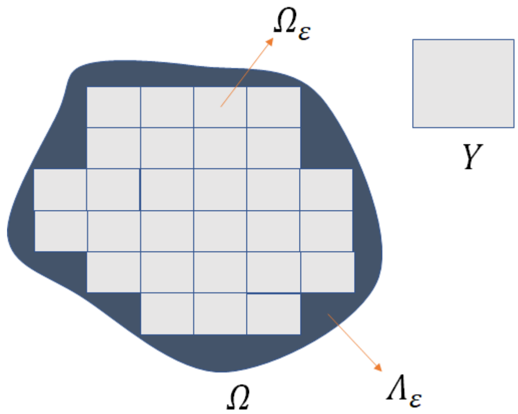


FIG. 1. *Tiling of* $\Omega$ *in the Euclidean set up.*

$y \in Y$. We now introduce this scale at the $\varepsilon$ level itself using the scale decomposition of the Euclidean space $\mathbb{R}^n$. We will later give an appropriate scale decomposition of the Heisenberg group. For $x \in \mathbb{R}^n$, we can write the $\varepsilon$-scale decomposition as $x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right)$. We introduce the scale $y$ for varying $\left\{\frac{x}{\varepsilon}\right\}_Y$ and we have the following definition.

DEFINITION 2 (unfolding operator). *For a* $\phi$ *Lebesgue-measurable real valued function on* $\Omega$*, the unfolding operator* $T^\varepsilon$ *is defined as follows:*

$$T^\varepsilon(\phi)(x,y) = \begin{cases} \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for } (x,y) \in \Omega_\varepsilon \times Y, \\ 0 & \text{for } (x,y) \in \Lambda_\varepsilon \times Y. \end{cases} \tag{1.2}$$

The test functions used in two scale convergence have one macroscale $x$ which tells the position in $\Omega$; another is microscale $\frac{x}{\varepsilon}$ which tells the position of $x$ in the reference cell. In unfolding this concept is used very explicitly. More precisely, if we take the domain as $\Omega = \bigcup_{k \in E_\varepsilon} \{k\varepsilon + \varepsilon Y\}$ and $\psi \in C_c^\infty(\Omega; C_\#^\infty(Y))$, then, we can write (1.1) as

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\psi^\varepsilon(x)\,dx \\
&= \lim_{\varepsilon\to 0} \sum_k \int_{k\varepsilon+\varepsilon Y} u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)dx = \lim_{\varepsilon\to 0} \sum_k \int_{k\varepsilon+\varepsilon Y} u_\varepsilon(k\varepsilon+\varepsilon y)\psi(k\varepsilon+\varepsilon y, y)\,dy \\
&= \frac{1}{|Y|} \lim_{\varepsilon\to 0} \int_\Omega \int_Y u_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right)\psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, y\right)dxdy \\
&= \frac{1}{|Y|} \lim_{\varepsilon\to 0} \int_{\Omega\times Y} T^\varepsilon(u_\varepsilon)(x,y)T^\varepsilon(\psi^\varepsilon)(x,y)\,dxdy = \frac{1}{|Y|}\int_{\Omega\times Y} u_0(x,y)\psi(x,y)\,dxdy.
\end{aligned}$$

Observe that the definition of two-scale convergence is reduced to weak convergence in $L^2(\Omega \times Y)$, and it is easy to apply as it is technically less demanding.

There are some advantages to using this method; for example, while doing optimal control problems in a periodic setup, the optimal control is easily characterized by the unfolding of the adjoint state, which helps to analyze asymptotic behavior; see

[2, 3, 23, 25, 26]. This method reduces the definition of two-scale convergence in $L^p(\Omega)$ to weak convergence of the unfolded sequence in $L^p(\Omega \times Y)$ for $1 < p < \infty$. This is a very effective method in analyzing the various multiscale problems; for details, see [1, 17] and references therein. The unfolding method in $\mathbb{R}^n$ is intensely dependent on the group structure of $\mathbb{R}^n$. We aim to develop a similar type of unfolding operator for the Heisenberg group. As we have already mentioned the concept of periodic functions and tiling in the Heisenberg group $\mathbb{H}^1$ was introduced in [9], which motivates us to define the greatest integer part for $x \in \mathbb{H}^1$, that is, $[x]_{\mathbb{H}}$ and fractional $\{x\}_{\mathbb{H}}$. Using these definitions, we have defined unfolding operator $T^\varepsilon$ in the Heisenberg group. The definition of $T^\varepsilon$ for $\mathbb{H}^1$ keeps periodic functions unchanged. As an application of this unfolding operator, we have considered two homogenization problems. First, we homogenize a PDE $-\text{div}_{\mathbb{H}}(A^\varepsilon \nabla_{\mathbb{H}})$ with Heisenberg-periodic oscillating coefficients in an open bounded domain $\Omega \subset \mathbb{H}^1$. This model PDE is also considered in [13], in a perforated domain where they have used two-scale convergence to analyze the asymptotic behavior.

To demonstrate the periodic unfolding method's applicability, we homogenize an interior optimal control problem with highly contrasting diffusivity coefficients. The homogenization of PDEs with high contrasting diffusivity coefficients in a Euclidean domain is a very interesting and useful topic; for example, see, [6, 11, 24, 28]. In these articles, the homogenization procedure was performed using an extension operator. But this kind of extension operator is unavailable in the Heisenberg group. Recently in an interesting work [25], the present authors have analyzed the asymptotic behavior of an optimal control problem with a high contrasting diffusivity coefficient in an oscillating domain without using an extension operator but with the help of a modified unfolding operator. In the present case, we will also use the periodic unfolding operator for the Heisenberg group for the problem under consideration.

The rest of this article is organized as follows. In section 2, we recall the definition and properties of periodic and nonperiodic function spaces in the Heisenberg group. In section 3, the definition of $[x]_{\mathbb{H}}$, $\{x\}_{\mathbb{H}}$, unfolding operators, and adjoint operators are introduced. Properties and their proof are also given here. Finally, in section 4, we consider a model PDE with oscillating coefficients, homogenize it, and also show the characterization of the interior periodic optimal control for the interior periodic optimal control problem. We did not present the final homogenization of the optimal control problem as it follows along similar lines. Moreover, we have also applied this unfolding operator to homogenize an optimal control problem with high contrasting diffusivity coefficients.

## 2. Preliminaries.

Here, we introduce required notations which will be used throughout the article and some preliminaries. We denote the 1-dimensional Heisenberg group by $\mathbb{H}^1 \cong \mathbb{R}^3$ and a typical point in $\mathbb{H}^1$ is denoted by $x = (x_1, x_2, x_3)$. For $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbb{H}^1$, the group operation is $p \cdot q = (p_1 + q_1, p_2 + q_2, p_3 + q_3 + 2(p_2 q_1 - p_1 q_2))$. The inverse of $x \in \mathbb{H}^1$ is $x^{-1} = (-x_1, -x_2, -x_3)$. The family of non-isotropic dilations are denoted by $\delta_\lambda$, defined as $\delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$ for $x \in \mathbb{H}^1$. The left translation operator corresponding to $p \in \mathbb{H}^1$ is denoted by $\tau_p$ and defined as $\tau_p(x) = p \cdot x$ for $x \in \mathbb{H}^1$. We consider the following homogeneous norm with respect to $\delta$; for $x \in \mathbb{H}^1$, $\|x\|_\infty := max\{\sqrt{x_1^2 + x_2^2},\ \sqrt{|x_3|}\}$. The associated distance between any $p, q \in \mathbb{H}^1$ is given as $d(p,q) = \|p^{-1} \cdot q\|_\infty$. There is a relation between this distance and Euclidean distance (see [13]), which is stated in the following proposition.

Proposition 3. *The function $d$ is a distance in $\mathbb{H}^1$. Further, it is homogeneous and left translation invariant, that is, for any $p,q,x\in\mathbb{H}^1$ and $\lambda>0$,*

$$d(\delta_\lambda q,\delta_\lambda x)=\lambda d(q,x) \quad \textit{and} \quad d(\tau_p q,\tau_p x)=d(q,x).$$

*For any bounded subset $\Omega$ of $\mathbb{H}^1$, there exist positive constants $c_1(\Omega),c_2(\Omega)$ such that*

$$c_1(\Omega)|p-q|_{\mathbb{R}^3}\le d(p,q)\le c_2(\Omega)\sqrt{|p-q|_{\mathbb{R}^3}}.$$

*Here $|\cdot|_{\mathbb{R}^3}$ denotes the Euclidean norm.*

In particular, the induced topology by $d$ and the Euclidean topology coincide on $\mathbb{H}^1$. The standard Lebesgue measure is the left and right invariant Haar measure for the group. For any measurable set $S\subset\mathbb{H}^1$, the Lebesgue measure of $S$ is denoted by $|S|$. Because of the anisotropic dilation $\delta_\lambda$ for $\lambda>0$, we have $|\delta_\lambda(S)|=\lambda^4|S|$. That is why the vector space dimension of $\mathbb{H}^1$ is 3, but the Hausdorff dimension is 4.

The Lie algebra of the left invariant vector fields of $\mathbb{H}^1$ is given by

$$X_1=\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_3},\quad X_2=\frac{\partial}{\partial x_2}-2x_1\frac{\partial}{\partial x_3},\ \text{and } X_3=\frac{\partial}{\partial x_3}.$$

The only nontrivial commutator relation is, $[X_1,X_2]=X_1X_2-X_2X_1=-4X_3$. The horizontal bundle $\mathrm{H}\mathbb{H}^1$ is the span of the vector field $\{X_1,X_2\}$. Hence, we will identify a section $\phi$ of $\mathrm{H}\mathbb{H}^1$ with the function $\phi=(\phi_1,\phi_2):\mathbb{H}^1\to\mathbb{R}^2$.

**Function spaces (see [13]).** Throughout this article, $\Omega\subset\mathbb{H}^1$ is a bounded domain. For any integer $k>0$, $C^k(\Omega)$, $C^\infty(\Omega)$ denote the usual differentiable function spaces in the Euclidean sense. We denote by $C^k(\Omega;\mathrm{H}\mathbb{H}^1)$, for $k\ge 0$, the set of all $C^k$ sections of $\mathrm{H}\mathbb{H}^1$. Now, we will define gradient and divergence as follows.

Definition 4. *Let $f\in C^1(\Omega)$ and $\phi=(\phi_1,\phi_2)\in C^1(\Omega;H\mathbb{H}^1)$ be a continuously differentiable section of $\mathrm{H}\mathbb{H}^1$. Define $\nabla_{\mathbb{H}}f:=(X_1f,X_2f)$ and $\mathrm{div}_{\mathbb{H}}\phi=X_1\phi_1+X_2\phi_2$.*

Note that both $\nabla_{\mathbb{H}}$, $\mathrm{div}_{\mathbb{H}}$ are left invariant differential operators. Also, $\nabla_{\mathbb{H}}f$ can be defined as a section of $\mathrm{H}\mathbb{H}^1$ as $\nabla_{\mathbb{H}}f=(X_1f)X_1+(X_2f)X_2$. The Heisenberg gradient $\nabla_{\mathbb{H}}$ can be written in terms of Euclidean gradient $\nabla$ as

$$\nabla_{\mathbb{H}}=C(x)\nabla,\ \text{where } C=C(x)=\begin{bmatrix}1 & 0 & 2x_2\\ 0 & 1 & -2x_1\end{bmatrix}.$$

Similarly, $\mathrm{div}_{\mathbb{H}}\phi=\mathrm{div}(C^t\phi)$, where div is the Euclidean divergence in $\mathbb{R}^3$ and $C^t$ is the transpose of the matrix $C$.

For $1\le p<\infty$, $L^p(\Omega)$ denotes the usual Euclidean $p$-integrable space. Here, we will introduce all the necessary nonperiodic function spaces:

(i) The set of all smooth sections of $\mathrm{H}\mathbb{H}^1$ is denoted by $C^\infty(\Omega;\mathrm{H}\mathbb{H}^1)$. Similarly the compactly supported smooth sections of $\mathrm{H}\mathbb{H}^1$ are denoted by $C_c^\infty(\Omega;\mathrm{H}\mathbb{H}^1)$.

(ii) Analogously to a standard Euclidean $H^1(\Omega)$ Sobolev space, we have the following Heisenberg Sobolev spaces, $H^1_{\mathbb{H}}(\Omega)=\{f\in L^2(\Omega):\ X_1f,X_2f\in L^2(\Omega)\}$. Further, $C^\infty(\Omega)\cap H^1_{\mathbb{H}}(\Omega)$ is dense in $H^1_{\mathbb{H}}(\Omega)$.

(iii) The closure of $C_c^\infty(\Omega)$ in $H^1_{\mathbb{H}}(\Omega)$ is denoted by $H^1_{0,\mathbb{H}}(\Omega)$.

Throughout this article, we will denote the cube $[0,2]^3$ by $Y$. We use this cube of side length 2 instead of $[0,1]^3$ to avoid the intersection of tiles in the Heisenberg periodic setting. A set $G\subset\mathbb{H}^1$ is said to be $Y$-periodic if for any $x\in\mathbb{H}^1$ and $k\in\mathbb{Z}^3$, the translations $\tau_{2k}(x)\in G$. The space $\mathbb{H}^1$ is indeed $Y$-periodic. In this article, we will use $\mathbb{H}^1$ as a $Y$-periodic set just like $\mathbb{R}^n=\biguplus_{k\in\mathbb{Z}^n}([0,1)^n+k)$.

(i) *Periodic function:* Let $f$ be a real valued function defined on $\mathbb{H}^1$. The function $f$ is said to be $Y$-periodic if for any $k \in \mathbb{Z}^3$, $f(\tau_{2k}(x)) = f(x)$ for all $x \in \mathbb{H}^1$. A section $\phi$ in $\mathrm{H}\mathbb{H}^1$ is $Y$-periodic if the canonical coordinates are $Y$-periodic.

(ii) We denote $C^\infty_{\#,\mathbb{H}}(Y)$ as the space of smooth real valued $Y$-periodic functions.

(iii) For $1 \le p < \infty$, we denote by $L^p_{\#,\mathbb{H}}(Y)$, the space of $Y$-periodic functions such that $f|_Y \in L^p(Y)$ endowed with the norm $\|f\|_{L^p(Y)}$.

(iv) Similarly, $H^1_{\#,\mathbb{H}}(Y)$ denotes the space of all $f \in L^2_{\#,\mathbb{H}}$ such that $X_i f \in L^2(\delta_\lambda(Y))$ for all $\lambda > 0$ endowed with the norm $\|f\|_{H^1_{\mathbb{H}}(Y)}$.

We now introduce the periodic vector valued function spaces:

(i) We denote by $C^\infty_c(\Omega; C^\infty_{\#,\mathbb{H}}(Y))$, the space of all smooth functions on $\Omega \times \mathbb{H}^1$ such that for any $\phi \in C^\infty_c(\Omega; C^\infty_{\#,\mathbb{H}}(Y))$, $x \to \phi(x, \cdot)$ is $C^\infty$ from $\Omega \to C^\infty_{\#,\mathbb{H}}(Y)$ with compact support.

(ii) The space of periodic smooth sections is denoted by $C^\infty_{\#,\mathbb{H}}(Y; \mathrm{H}\mathbb{H}^1)$.

(iii) The space $H^1_{\#}(Y; \mathrm{H}\mathbb{H}^1)$ is defined as the set $\{\phi = (\phi_1, \phi_2)\}$ of all measurable sections of $\mathrm{H}\mathbb{H}^1$ such that $\phi \in (H^1_{\#,\mathbb{H}}(Y))^2$.

(iv) The space $V^{\mathrm{div}}_{\#,\mathbb{H}}(Y)$ is the completion of $\{u \in C^\infty_{\#,\mathbb{H}}(Y; \mathrm{H}\mathbb{H}^1)\}$ with respect to the following norm, $\|u\|_{V^{\mathrm{div}}_{\#,\mathbb{H}}(Y)} = \|u\|_{L^2_{\#,\mathbb{H}}(Y;\mathrm{H}\mathbb{H}^1)} + \|\mathrm{div}_{\mathbb{H}} u\|_{L^2_{\#,\mathbb{H}}(Y)}$.

We now state a version of Theorem 2.16 from [13], which will be used in our analysis.

THEOREM 5. *Let $F \in L^2(\Omega; V^{\mathrm{div}}_{\#,\mathbb{H}}(Y)^*)$ such that*

$$\int_\Omega \langle F(x), \phi(x, \cdot)\rangle_{V^{div}_{\#,\mathbb{H}}(Y)^*, V^{div}_{\#,\mathbb{H}}(Y)} dx = 0$$

*for all $\phi(x,y) \in L^2(\Omega; V^{\mathrm{div}}_{\#,\mathbb{H}}(Y))$ with $\mathrm{div}_{\mathbb{H},y}\phi = 0$ for a.e $x \in \Omega$. Then, $F = \nabla_{\mathbb{H},y}\psi$ with $\psi \in L^2(\Omega; L^2_{\#,\mathbb{H}}(Y)/\mathbb{R})$.*

**3. Definition and properties of unfolding operator.** It has been proved in [9] for $Y = [-1,1)^3$ that there is a canonical tiling of $\mathbb{H}^1$ associated with the structure of $\mathbb{H}^1$ as a group with dilations, defined as follows.

DEFINITION 6. *Let $\varepsilon > 0$ be fixed. Let a typical point of $\mathbb{Z}^3$ be denoted by $k = (k_1, k_2, k_3)$. Define $Y^\varepsilon_k = \delta_\varepsilon(2k \cdot Y)$. Then $\{Y^\varepsilon_k : k \in \mathbb{Z}^3\}$ is a tiling of $\mathbb{H}^1$, i.e., $Y^\varepsilon_k \cap Y^\varepsilon_h = \phi$ if $k \neq h$ and $\mathbb{H}^1 = \bigcup_{k\in\mathbb{Z}^3} Y^\varepsilon_k$.*

The above tilling also holds for $Y = [0,2)^3$. We will use the tiling of $\Omega$ with $Y = [0,2)^3$. A slightly modified definition of greatest integer function will be used. Let $x \in \mathbb{H}^1$, then $x \in 2k \cdot Y$ for some $k \in \mathbb{Z}^3$. Here we can write $x = 2k \cdot y$ for some $y \in Y$. Thus, we have $x_1 = 2k_1 + y_1$, $x_2 = 2k_2 + y_2$, $x_3 = 2k_3 + y_3 + 4(k_2 y_1 - k_1 y_2)$. It shows that $2k_1$, $2k_2$, and $2k_3$ are the greatest even integers less than $x_1, x_2$, and $x_3 - 4k_2 y_1 + 4y_1 k_2$. This leads us to define the greatest even integer function. For any $r \in \mathbb{R}$ define $[r]_e =$ greatest even integer less than or equal to $r$. An analogous definition of the even fractional part is $\{r\}_e = r - [r]_e$. For any $x \in \mathbb{H}^1$, define

$$[x]_{\mathbb{H}} = \frac{1}{2}\left([x_1]_e, [x_2]_e, [x_3 - 2([x_2]_e\{x_1\}_e - [x_1]_e\{x_2\}_e)]_e\right).$$

The fractional part of $x$ in $\mathbb{H}^1$ is defined by

$$\begin{aligned}\{x\}_{\mathbb{H}} = 2[x]^{-1}_{\mathbb{H}} \cdot x = (&x_1 - [x_1]_e, x_2 - [x_2]_e,\\ &x_3 - [x_3 - 2([x_2]_e\{x_1\}_e - [x_1]_e\{x_2\}_e)]_e - 2([x_2]_e x_1 - [x_1]_e x_2))\\ = (&\{x_1\}_e, \{x_2\}_e, x_3 - [x_3 - 2([x_2]_e\{x_1\}_e - [x_1]_e\{x_2\}_e)]_e - 2([x_2]_e x_1 - [x_1]_e x_2)).\end{aligned}$$

Now, note that $x_i = [x_i]_e + \{x_i\}_e$ for $i = 1, 2$. Using this identity, we have

$$\begin{aligned} 2([x_2]_e x_1 - [x_1]_e x_2) &= 2([x_2]_e[x_1]_e + [x_2]_e\{x_1\}_e) - [x_1]_e[x_2]_e - [x_1]_e\{x_2\}_e \\ &= 2([x_2]_e\{x_1\}_e - [x_1]_e\{x_2\}_e). \end{aligned}$$

Hence, for $x \in \mathbb{H}^1$, the definition for the fractional part can be rewritten as

$$\begin{aligned} \{x\}_{\mathbb{H}} = (\{x_1\}_e, \{x_2\}_e, x_3 - [x_3 - 2([x_2]_e\{x_1\}_e - [x_1]_e\{x_2\}_e)]_e \\ - 2([x_2]_e\{x_1\}_e - [x_1]_e\{x_2\}_e)). \end{aligned}$$

Now for any $x \in Y_k^\varepsilon$, we can recover $k$ from $x$ as

$$k = (k_1, k_2, k_3) = \frac{1}{2}\left(\left[\frac{x_1}{\varepsilon}\right]_e, \left[\frac{x_2}{\varepsilon}\right]_e, \left[\frac{x_3}{\varepsilon^2} - 2[x_2/\varepsilon]_e\{x_1/\varepsilon\}_e + 2[x_1/\varepsilon]_e\{x_2/\varepsilon\}_e\right]_e\right).$$

For any $x \in \mathbb{H}$, using the definition of $[\,]_{\mathbb{H}}$ and $\{\}_{\mathbb{H}}$, we have

$$x = 2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}} \cdot \delta_\varepsilon\left\{\delta_{\frac{1}{\varepsilon}}x\right\}_{\mathbb{H}} = \delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}} \cdot \left\{\delta_{\frac{1}{\varepsilon}}x\right\}_{\mathbb{H}}\right) = \delta_\varepsilon\left(2k \cdot \left\{\delta_{\frac{1}{\varepsilon}}x\right\}_{\mathbb{H}}\right). \tag{3.1}$$

Let $\varepsilon > 0$, and $\Omega \subset \mathbb{H}^1$ be a bounded domain. Let $E_\varepsilon = \{k \in \mathbb{Z}^3 : Y_k^\varepsilon \subset \Omega\}$, $\Omega_\varepsilon = \bigcup_{k \in E_\varepsilon} Y_k^\varepsilon$, and $\Lambda_\varepsilon = \Omega \backslash \Omega_\varepsilon$. Now with the above notations, we are in the position to define the unfolding operator in the context of a Heisenberg group.

Definition 7 (unfolding operator). *Let $\varepsilon > 0$, then the $\varepsilon$-unfolding of a function $\phi : \Omega \to \mathbb{R}$ is the function $T^\varepsilon\phi : \Omega \times Y \to \mathbb{R}$ defined as*

$$T^\varepsilon(\phi)(x, y) = \begin{cases} \phi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\right) \cdot \delta_\varepsilon y\right) & \text{for } (x, y) \in \Omega_\varepsilon \times Y, \\ 0 \quad \text{for } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

The operator $T^\varepsilon$ is the unfolding operator; some important properties are given below.

Proposition 8. *Let the unfolding operator $T^\varepsilon$ be defined as above, then $T^\varepsilon$ is linear and for $\phi_1, \phi_2 : \Omega \to \mathbb{R}$, $T^\varepsilon(\phi_1\phi_2) = T^\varepsilon(\phi_1)T^\varepsilon(\phi_2)$.*

This follows directly and the important $L^1$ integral identity is proved below.

Proposition 9. *Let $\phi \in L^1(\Omega)$. Then, $\displaystyle\int_{\Omega_\varepsilon} \phi\, dx = \frac{1}{|Y|}\int_{\Omega \times Y} T^\varepsilon(\phi)\, dxdy$.*

*Proof.* We have $\displaystyle\frac{1}{|Y|}\int_{\Omega \times Y} T^\varepsilon(\phi)\, dxdy$

$$\begin{aligned} &= \frac{1}{|Y|}\int_\Omega\int_Y \phi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\right) \cdot \delta_\varepsilon y\right) dxdy = \sum_{k \in E_\varepsilon} \frac{1}{|Y|}\int_{Y_k^\varepsilon}\int_Y \phi(\delta_\varepsilon(2k) \cdot \delta_\varepsilon y)\, dxdy \\ &= \sum_{k \in E_\varepsilon} \frac{1}{|Y|}\int_Y \phi(\delta_\varepsilon(2k) \cdot \delta_\varepsilon y)|Y_k^\varepsilon| dy = \sum_{k \in E_\varepsilon} \frac{1}{|Y|}\int_Y \phi(\delta_\varepsilon(2k) \cdot \delta_\varepsilon y)\varepsilon^4|Y| dy. \end{aligned}$$

We make the following change of variables as

$$z_1 = \varepsilon(2k_1 + y_1),\ z_2 = \varepsilon(2k_2 + y_2),\ z_3 = \varepsilon^2(2k_3 + y_3 + 4(k_2y_1 - k_1y_2)).$$

We have $dz = \varepsilon^4 dy$. By the above change of variables, we have

$$\frac{1}{|Y|}\int_{\Omega \times Y} T^\varepsilon(\phi)\, dxdy = \sum_{k \in E_\varepsilon}\int_{Y_k^\varepsilon} \phi(z) dz = \int_{\Omega_\varepsilon} \phi\, dx.$$ ∎

The above integral identity gives us the following proposition.

PROPOSITION 10.

(i) *For* $p \in [1,\infty)$*, the operator* $T^\varepsilon$ *is linear continuous from* $L^p(\Omega)$ *to* $L^p(\Omega\times Y)$*.*

(ii) $\displaystyle \frac{1}{|Y|}\int_{\Omega\times Y} T^\varepsilon(\phi)dxdy = \int_\Omega \phi dx - \int_{\Lambda_\varepsilon}\phi dx = \int_{\Omega_\varepsilon}\phi dx.$

(iii) $\left|\int_\Omega \phi dx - \frac{1}{|Y|}\int_{\Omega\times Y} T^\varepsilon(\phi)\, dxdy\right| \le \int_{\Lambda_\varepsilon}|\phi|dx.$

(iv) $\|T^\varepsilon(\phi)\|_{L^p(\Omega\times Y)} \le |Y|^{\frac{1}{p}}\|\phi\|_{L^p(\Omega)}.$

Here, we are considering the domain as a bounded open subset of $\mathbb{H}^1$. Since the Hausdorff dimension of $\mathbb{H}^1$ is 4, then the cardinality of the set

$$\{k \in \mathbb{Z}^3 : Y_k^\varepsilon \cap \partial\Omega \text{ is nonempty}\}$$

is $O\left(\frac{1}{\varepsilon^3}\right)$. Hence $|\Lambda_\varepsilon| = O(\varepsilon)$. Also $\chi_{\Lambda_\varepsilon} \to 0$ pointwise as $\varepsilon \to 0$. Hence, we have the following proposition.

PROPOSITION 11. *Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $L^p(\Omega)$ *with* $p \in (1,\infty)$ *and* $v \in L^q(\Omega)$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, then* $\int_{\Lambda_\varepsilon} u_\varepsilon v\, dx \to 0$ *as* $\varepsilon \to 0$*.*

*Proof.* Since $\chi_{\Lambda_\varepsilon} \to 0$ as $\varepsilon \to 0$, by the Lebesgue dominated convergence theorem, we get $\int_\Omega \chi_{\Lambda_\varepsilon}|v|^q\, dx \to 0$, and by Holder's inequality, we have $\int_{\Lambda_\varepsilon} u_\varepsilon v\, dx \to 0$. ☐

*Remark:* In the theory of homogenization or multiscale analysis the final goal is to pass to the limit as $\varepsilon \to 0$. If functions in some integral satisfy the hypothesis of the Proposition 11, say, for example, $u_\varepsilon$ and $v$ are as in Proposition 11, then, we use the following convention, $\int_\Omega u_\varepsilon v = \frac{1}{|Y|}\int_{\Omega\times Y} T^\varepsilon(u_\varepsilon)T^\varepsilon(v)$, that is instead of writing approximate equality, we choose to write equality since at the end, we will pass to the limit $\varepsilon \to 0$.

LEMMA 12. *Let* $\phi \in C_c^\infty(\Omega)$*. Then,* $\|T^\varepsilon(\phi) - \phi\|_\infty \to 0$ *in* $\Omega\times Y$ *as* $\varepsilon \to 0$

*Proof.* Since $\phi$ is a compactly supported smooth function, it is Lipschitz, let's say with Lipschitz constant $L$. Fix $x \in \Omega_\varepsilon$. Then

$$\begin{aligned} |T^\varepsilon(\phi)(x,y) - \phi(x)| &= \left|\phi\left(2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\cdot\delta_\varepsilon y\right) - \phi(x)\right| \le L\left|x - 2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\cdot\delta_\varepsilon y\right|_{\mathbb{R}^3} \\ &\le C\, d\left(2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\cdot\delta_\varepsilon\left\{\delta_{\frac{1}{\varepsilon}}x\right\},\, 2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\cdot\delta_\varepsilon y\right) \\ &\le Cd\left(\delta_\varepsilon\left\{\delta_{\frac{1}{\varepsilon}}x\right\}_{\mathbb{H}},\, \delta_\varepsilon y\right) \le C\varepsilon. \end{aligned}$$

The last two inequalities follow from Proposition 3 and hence the result follows as $\varepsilon \to 0$. ☐

Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, the above lemma leads to the following.

LEMMA 13. *For* $v \in L^2(\Omega)$*, we have* $T^\varepsilon(v) \to v$ *strongly in* $L^2(\Omega\times Y)$*.*

We recall the definition of two-scale convergence for the Heisenberg group [13].

DEFINITION 14. *A family of functions* $\{u_\varepsilon\} \in L^2(\Omega)$ *is said to be two-scale convergent in* $\mathbb{H}^1$ *to* $u_0 \in L^2(\Omega\times Y)$*, if for any* $\psi \in C_c^\infty(\Omega; C_{\#.\mathbb{H}}^\infty(Y))$*, we have*

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\psi\left(x, \delta_{\frac{1}{\varepsilon}}(x)\right)dx = \frac{1}{|Y|}\int_{\Omega\times Y} u_0(x,y)\psi(x,y)\, dxdy. \tag{3.2}$$

We have already discussed in the introduction that two-scale convergence of a sequence in $L^2(\Omega)$ is equivalent to weak convergence of the unfolded sequence in $L^2(\Omega\times Y)$. This result also holds in the Heisenberg group, which is stated in the following proposition.

PROPOSITION 15. *Let $\{v_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then the following statements are equivalent:*

1. *$v_\varepsilon$ two-scale converges in $\mathbb{H}^1$ to $v_0 \in L^2(\Omega \times Y)$.*
2. *$T^\varepsilon(v_\varepsilon)$ weakly converges to $v_0 \in L^2(\Omega \times Y)$.*

*Proof.* The proof is based on Lemma 12. For $\phi \in C_c^\infty(\Omega; C^\infty_{\#,\mathbb{H}}(Y))$, for $\varepsilon > 0$ small enough, we have

$$\int_\Omega v_\varepsilon(x)\phi\left(x, \delta_{\frac{1}{\varepsilon}}(x)\right) dx = \int_{\Omega\times Y} T^\varepsilon(v_\varepsilon)\phi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\right)\cdot\delta_\varepsilon(y), y\right) dxdy + o(1). \tag{3.3}$$

Let $v_\varepsilon$ two-scale converge to $v_0$ in $\mathbb{H}^1$ and $T^\varepsilon v_\varepsilon \rightharpoonup \hat{v}_0$ weakly in $L^2(\Omega \times Y)$. By passing to $\varepsilon \to 0$ on both side of (3.3), we get

$$\frac{1}{|Y|}\int_{\Omega\times Y} v_0(x,y)\phi(x,y)\,dxdy = \frac{1}{|Y|}\int_{\Omega\times Y} \hat{v}_0(x,y)\phi(x,y)\,dxdy.$$

As $\phi \in C_c^\infty(\Omega; C^\infty_{\#,\mathbb{H}}Y)$ is arbitrary, it implies $v_0(x,y) = \hat{v}_0(x,y)$ a.e. in $\Omega \times Y$. ☐

**3.1. Averaging and adjoint operators.** Let $u \in L^p(\Omega)$ and $v \in L^q(\Omega \times Y)$. Then, we compute

$$\begin{aligned}\frac{1}{|Y|}\int_{\Omega\times Y} T^\varepsilon(u)(x,y)v(x,y)\,dxdy &= \frac{1}{|Y|}\int_{\Omega\times Y} u\left(2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\cdot\delta_\varepsilon y\right)v(x,y)\,dxdy\\ &= \sum_{k\in E_\varepsilon}\frac{1}{|Y|}\int_{Y_k^\varepsilon}\int_Y u\left(2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\cdot\delta_\varepsilon y\right)v(x,y)\,dxdy\\ &= \sum_{k\in E_\varepsilon}\frac{1}{|Y|}\int_Y\int_Y u((\delta_\varepsilon(2k)\cdot\delta_\varepsilon y))v(\delta_\varepsilon(2k)\cdot\delta_\varepsilon z, y)\varepsilon^4 dzdy.\end{aligned}$$

Applying the following change of variables

$$x_1 = \varepsilon(2k_1 + y_1),\ x_2 = \varepsilon(2k_2 + y_2),\ x_3 = \varepsilon^2(2k_3 + y_3 + 4(k_2 y_1 - y_2 k_1)),$$

we obtain $\frac{1}{|Y|}\int_{\Omega\times Y} T^\varepsilon(u)(x,y)v(x,y)\,dxdy$

$$\begin{aligned}&= \sum_{k\in E_\varepsilon}\frac{1}{|Y|}\int_{\delta_\varepsilon(2k\cdot Y)}\int_Y u(x)v(\delta_\varepsilon(2k)\cdot\delta_\varepsilon z, \{\delta_{\frac{1}{\varepsilon}}x\}_{\mathbb{H}})\,dxdz\\ &= \sum_{k\in E_\varepsilon}\int_{\delta_\varepsilon(2k\cdot Y)} u(x)\left(\frac{1}{|Y|}\int_Y v\left(2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\cdot\delta_\varepsilon z, \{\delta_{\frac{1}{\varepsilon}}(x)\}_{\mathbb{H}}\right)dz\right)dx\\ &= \int_\Omega u(x)\left(\frac{1}{|Y|}\int_Y v\left(2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\cdot\delta_\varepsilon z, \{\delta_{\frac{1}{\varepsilon}}(x)\}_{\mathbb{H}}\right)dz\right)dx.\end{aligned} \tag{3.4}$$

This motivates us to define the following.

DEFINITION 16. *For $p \in (1,\infty)$, the averaging operator $\mathcal{U}_\varepsilon : L^p(\Omega\times Y) \to L^p(\Omega)$ is defined as*

$$\mathcal{U}_\varepsilon(\phi)(x) = \begin{cases} \frac{1}{|Y|}\int_Y \phi\left(2\delta_\varepsilon\left[\delta_{\frac{1}{\varepsilon}}x\right]_{\mathbb{H}}\cdot\delta_\varepsilon z, \{\delta_{\frac{1}{\varepsilon}}(x)\}_{\mathbb{H}}\right)dz & a.e.\ x \in \Omega_\varepsilon,\\ 0 & a.e.\ x \in \Lambda_\varepsilon.\end{cases}$$

By (3.4), we have for $\psi \in L^p(\Omega)$ and $\phi \in L^q(\Omega \times Y)$,

$$\int_\Omega \mathcal{U}_\varepsilon(\phi)(x)\psi(x)\,dx = \frac{1}{|Y|}\int_{\Omega\times Y} \phi(x,y)T^\varepsilon(\psi)(x,y)\,dxdy.$$

Hence, this implies that the adjoint operator of $T^\varepsilon$ is $\mathcal{U}_\varepsilon$ in the above sense. Now, we will see how the unfolding operator behaves with gradients.

**3.2. Unfolding of the gradient.** Throughout this article, we will denote $\nabla_{\mathbb{H}}$ and $\nabla_{\mathbb{H},y}$, the gradient with respect to $x$ and $y$, respectively, on the Heisenberg group. Similarly, $\mathrm{div}_{\mathbb{H}}$ and $\mathrm{div}_{\mathbb{H},y}$ denote the divergence with respect to $x$ and $y$, respectively. Now, we will see the relation between $\nabla_{\mathbb{H}}$ and $\nabla_{\mathbb{H},y}$, and between $\mathrm{div}_{\mathbb{H}}$ and $\mathrm{div}_{\mathbb{H},y}$. Recall the horizontal vector fields

$$X_1 = \frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2} - 2x_1\frac{\partial}{\partial x_3},$$
$$Y_1 = \frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_3}, \quad Y_2 = \frac{\partial}{\partial y_2} - 2y_1\frac{\partial}{\partial y_3}.$$

Let $\phi \in H^1_{\mathbb{H}}(\Omega)$ and let $x \in Y^\varepsilon_k$. Then

$$T^\varepsilon(\phi)(x,y) = \phi(\varepsilon(2k_1+y_1), \varepsilon(2k_2+y_2), \varepsilon^2(2k_3+y_3+4(k_2y_1-k_1y_2)) = \phi(\delta_\varepsilon(2k\cdot y))$$

for any $y \in Y$. By applying the horizontal vector field $Y_1$ on $T^\varepsilon(\phi)$, we get

$$\begin{aligned} Y_1(T^\varepsilon(\phi)(x,y)) &= \left(\frac{\partial}{\partial y_1} + 2y_2\frac{\partial}{\partial y_3}\right)(\phi(\delta_\varepsilon(2k\cdot y))) \\ &= \varepsilon\frac{\partial\phi}{\partial x_1}(\delta_\varepsilon(2k\cdot y)) + 2\varepsilon^2(2k_2+y_2)\frac{\partial\phi}{\partial x_3}(\delta_\varepsilon(2k\cdot y)) \\ &= \varepsilon T^\varepsilon\left(\frac{\partial\phi}{\partial x_1}\right) + 2\varepsilon T^\varepsilon(x_2)T^\varepsilon\left(\frac{\partial\phi}{\partial x_3}\right)(x,y) = \varepsilon T^\varepsilon(X_1\phi)(x,y). \end{aligned}$$

Similarly, we have $Y_2(T^\varepsilon(\phi)(x,y)) = \varepsilon T^\varepsilon(X_2(\phi))$. Hence using these two relations, we get the following identities:

$$\begin{aligned} \nabla_{\mathbb{H},y}(T^\varepsilon(\phi)(x,y)) &= \varepsilon T^\varepsilon(\nabla_{\mathbb{H}}\phi)(x,y), \\ \mathrm{div}_{\mathbb{H},y}(T^\varepsilon(\phi)(x,y)) &= \varepsilon T^\varepsilon(\mathrm{div}_{\mathbb{H}}\phi)(x,y). \end{aligned} \tag{3.5}$$

THEOREM 17. *Let $\{u_\varepsilon\}$ be a sequence in $H^1_{\mathbb{H}}(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $H^1_{\mathbb{H}}(\Omega)$. Then there exists a unique $u_1 \in L^2(\Omega; H^1_{\#,\mathbb{H}}(Y)/\mathbb{R})$ such that*
(i) *$T^\varepsilon(u_\varepsilon) \to u$ strongly in $L^2(\Omega\times Y)$,*
(ii) *$T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon) \rightharpoonup \nabla_{\mathbb{H}}u + \nabla_{\mathbb{H},y}u_1$ weakly in $(L^2(\Omega\times Y))^2$.*

*Proof.* First, we will show that the limit of $T^\varepsilon(u_\varepsilon)$ is independent of $y$. Let $T^\varepsilon(u_\varepsilon) \rightharpoonup \hat{u}$ weakly in $L^2(\Omega\times Y)$ and we need to show that $\hat{u}(x,y) = \hat{u}(x)$. To see this, for $\psi \in C^\infty_c(\Omega\times Y))$, consider the following:

$$\begin{aligned} \int_{\Omega\times Y} \varepsilon T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)\psi(x,y)\,dxdy &= \int_{\Omega\times Y} \nabla_{\mathbb{H},y}(T^\varepsilon(u_\varepsilon)(x,y))\psi(x,y)\,dxdy \\ &= -\int_{\Omega\times Y} T^\varepsilon(u_\varepsilon)(x,y)\mathrm{div}_{\mathbb{H},y}\psi(x,y)\,dxdy. \end{aligned} \tag{3.6}$$

As $T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)$ is bounded in $L^2(\Omega\times Y)$, we have $\int_{\Omega\times Y}\varepsilon T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)\psi(x,y) \to 0$ as $\varepsilon \to 0$. As $\varepsilon \to 0$ in (3.6) we get $\int_{\Omega\times Y}\hat{u}(x,y)\mathrm{div}_{\mathbb{H},y}\psi(x,y) = 0$ for all $\psi \in C^\infty_c(\Omega\times Y)$.

Hence $\nabla_{\mathbb{H},y}\hat{u}(x,y) = 0$, implies that $\hat{u}$ is independent of $y$. On the other hand by weak convergence of $u_\varepsilon$, we have $u = M_Y(\hat{u}) = \hat{u}$, where $M_Y(\hat{u}) = \frac{1}{|Y|}\int_Y \hat{u}(x,y)dy$. So, we have the weak convergence of $T^\varepsilon(u_\varepsilon) \rightharpoonup u$ in $L^2(\Omega\times Y)$. Now for the norm convergence, consider

$$\int_{\Omega\times Y}(T^\varepsilon(u_\varepsilon) - T^\varepsilon(u))^2\,dx \le |Y|\int_\Omega (u_\varepsilon - u)^2\,dxdy \to 0\;as\;\varepsilon\to 0.$$

We know from Lemma 13, that $\|T^\varepsilon u\|_{L^2(\Omega\times Y)} \to \|u\|_{L^2(\Omega\times Y)}$. Thus, we have

$$\|T^\varepsilon(u_\varepsilon)\|_{L^2(\Omega\times Y)} \to \|u\|_{L^2(\Omega\times Y)}.$$

Hence weak convergence with norm convergence implies the strong convergence. This proves (i) of Theorem 17.

For the second part, we will use the test function of the form $\psi_\varepsilon(x) = \psi(x,\delta_{\frac{1}{\varepsilon}}(x))$ for $\psi\in(C_c^\infty(\Omega, C^\infty_{\#,\mathbb{H}}(Y)))^2$ with $\mathrm{div}_{\mathbb{H},y}\psi = 0$. Let us consider the following:,

$$\int_\Omega \nabla_{\mathbb{H}}u_\varepsilon\psi\left(x,\delta_{\frac{1}{\varepsilon}}(x)\right)\,dx = \int_{\Omega\times Y} T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)(x,y)\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right),y\right)\,dxdy.$$

Let $T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon) \rightharpoonup \xi_0$ weakly in $(L^2(\Omega\times Y))^2$. Now, using integration by parts and the relations between $\nabla_{\mathbb{H}}$ and $\nabla_{\mathbb{H},y}$, $\mathrm{div}_{\mathbb{H}}$ and $\mathrm{div}_{\mathbb{H},y}$ given in (3.5), we get

$$\begin{aligned}
&\int_{\Omega\times Y} T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)(x,y)\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right),y\right)\\
&\quad= \int_{\Omega\times Y}\frac{1}{\varepsilon}\nabla_{\mathbb{H},y}T^\varepsilon(u_\varepsilon)(x,y)\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right),y\right)\\
&\quad= -\int_{\Omega\times Y} T^\varepsilon(u_\varepsilon)(x,y)\left[\mathrm{div}_{\mathbb{H}}\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right),y\right)\right]\\
&\quad= -\int_{\Omega\times Y} T^\varepsilon(u_\varepsilon)(x,y)T^\varepsilon(\mathrm{div}_{\mathbb{H}}\psi)(x,y).
\end{aligned}$$

By passing to the limit on both sides, we get

$$\int_{\Omega\times Y}\xi_0(x,y)\psi(x,y) = -\int_{\Omega\times Y} u(x)\mathrm{div}_{\mathbb{H}}\psi(x,y).$$

Thus, we get $\int_{\Omega\times Y}(\xi_0 - \nabla_{\mathbb{H}}u(x))\psi(x,y)\,dxdy = 0$. Convergence of unfolding sequences implies, $\xi_0, \nabla_{\mathbb{H}}u\in(L^2(\Omega\times Y))^2$. Hence $(\xi_0(x,y) - \nabla_{\mathbb{H}}u(x))\in L^2(\Omega;(V^{\mathrm{div}}_{\#,\mathbb{H}}(Y))^*)$. Since $(C_c^\infty(\Omega;C^\infty_{\#,\mathbb{H}}(Y)))^2$ is dense in $L^2(\Omega;(V^{\mathrm{div}}_{\#,\mathbb{H}}(Y)))$, we get

$$\int_{\Omega\times Y}(\xi_0 - \nabla_{\mathbb{H}}u(x))\psi(x,y)\,dxdy = 0\;\text{ for all } \psi\in L^2\left(\Omega;\left(V^{\mathrm{div}}_{\#,\mathbb{H}}(Y)\right)\right)\text{ with } \mathrm{div}_{\mathbb{H}}\psi = 0.$$

Hence, $(\xi_0 - \nabla_{\mathbb{H}}u)$ is perpendicular to the divergence free vector field. We get from Theorem 5 that there exists a unique $u_1\in L^2(\Omega, L^2_{\#,\mathbb{H}}(Y)/\mathbb{R})$ such that

$$\xi_0 - \nabla_{\mathbb{H}}u = \nabla_{\mathbb{H},y}u_1.$$

Since $\xi_0$ and $\nabla_{\mathbb{H}}u$ are in $(L^2(\Omega\times Y))^2$, we see that $u_1\in L^2(\Omega;H^1_{\#,\mathbb{H}}(Y)/\mathbb{R})$. Hence, we have the second convergence. ∎

The unfolding $T^\varepsilon$ exhibits more nice properties, which are useful in applications.

PROPOSITION 18. *Let $u_\varepsilon$ be a bounded sequence in $L^p(\Omega)$ with $p\in(1,\infty)$ satisfying*

$$\varepsilon\,\|X_i u_\varepsilon\|_{L^p(\Omega)} \le C \quad \text{for } i=1,2.$$

*Then there exists a subsequence and $\hat{u}\in L^p(\Omega)$ with $Y_i\hat{u}\in L^p(\Omega\times Y)$ such that*

$$\begin{aligned}&\text{(i) } T^\varepsilon(u_\varepsilon)\rightharpoonup \hat{u} \ \ \textit{weakly in } L^p(\Omega\times Y),\\ &\text{(ii) } \varepsilon T^\varepsilon(X_i u_\varepsilon)=Y_i T^\varepsilon(u_\varepsilon)\rightharpoonup Y_i\hat{u} \ \ \textit{weakly in } \ L^p(\Omega\times Y) \text{ for } i=1,2.\end{aligned} \tag{3.7}$$

*Proof.* As $u_\varepsilon$ is a bounded sequence in $L^p(\Omega)$, by properties of unfolding operator, we have $T^\varepsilon(u_\varepsilon)$ is a bounded sequence in $L^p(\Omega\times Y)$. Hence by weak compactness, there exists $\hat{u}\in L^p(\Omega\times Y)$ such that

$$T^\varepsilon(u_\varepsilon)\rightharpoonup \hat{u} \quad \text{weakly in } L^p(\Omega\times Y).$$

Let $\phi\in C_c^\infty(\Omega\times Y)$, consider

$$\int_{\Omega\times Y}\varepsilon T^\varepsilon(X_i u_\varepsilon)\phi\,dxdy=\int_{\Omega\times Y}Y_iT^\varepsilon(u_\varepsilon)\phi\,dxdy=-\int_{\Omega\times Y}T^\varepsilon(u_\varepsilon)Y_i\phi\,dxdy.$$

Using the weak convergence of $T^\varepsilon(u_\varepsilon)$, we pass to the limit as $\varepsilon\to 0$ to get

$$\lim_{\varepsilon\to 0}\int_{\Omega\times Y}\varepsilon T^\varepsilon(X_i u_\varepsilon)\phi\,dxdy=-\int_{\Omega\times Y}\hat{u}Y_i\phi\,dxdy.$$

The above equality implies the second part of the proposition. ∎

The above proposition can be written in the following form.

PROPOSITION 19. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$ with $p\in(1,\infty)$ satisfying*

$$\varepsilon\,\|\nabla_{\mathbb{H}} u_\varepsilon\|_{L^p(\Omega)}\le C.$$

*Then, there exists a subsequence and $\hat{u}\in L^p(\Omega;H^1_{\mathbb{H}}(Y))$ such that*

$$\begin{aligned}&T^\varepsilon(u_\varepsilon)\rightharpoonup \hat{u} \ \ \textit{weakly in } L^p(\Omega;H^1_{\mathbb{H}}(Y)),\\ &\varepsilon T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)=\nabla_{\mathbb{H},y}T^\varepsilon(u_\varepsilon)\rightharpoonup\nabla_{\mathbb{H},y}\hat{u} \ \ \textit{weakly in } \ (L^p(\Omega\times Y))^2.\end{aligned} \tag{3.8}$$

Now, let $u_\varepsilon$ be a sequence in $H^1_{\mathbb{H}}(\Omega)$ which weakly converges to $u$ in $H^1_{\mathbb{H}}(\Omega)$. Then, by compact embedding, $u_\varepsilon\to u$ strongly in $L^2(\Omega)$. By the properties of the unfolding operator, we have $T^\varepsilon(u_\varepsilon)\to u$ strongly in $L^2(\Omega\times Y)$. Thus, we have the following proposition.

PROPOSITION 20. *Let $\{u_\varepsilon\}$ be a sequence in $H^1_{\mathbb{H}}(\Omega)$ weakly convergent to $u$ in $H^1_{\mathbb{H}}(\Omega)$. Then $T^\varepsilon(u_\varepsilon)\to u$* weakly in $L^2(\Omega;H^1_{\mathbb{H}}(Y))$, and strongly in $L^2(\Omega\times Y)$.

**4. Homogenization via a periodic unfolding operator.** In this section, we study the homogenization of two problems, namely, a standard oscillation problem in the Heisenberg group and the homogenization of an optimal control problem with high contrasting diffusivity coefficients. An homogenization problem with high contrasting coefficients together with high oscillations is itself challenging; see our recent articles [24, 25] in this direction. See also [28]. We are investigating the applicability of the introduced unfolding operator in particular to optimal control problems in the Heisenberg group. At this stage, we would like to recall that the unfolding operator can be used to characterize the optimal control in the homogenization problem (see [2, 23, 26]) in the Euclidean setup.

Let $A=[a_{i,j}]_{i,j=1}^2 : \mathbb{H}^1 \to M_{2\times 2}(\mathbb{R})$ be a matrix valued function with the following properties:

1. The coefficients $a_{i,j} : \mathbb{H}^1 \to \mathbb{R}$ are Heisenberg $Y$-periodic for all $i,j=1,2$, bounded and measurable functions.
2. The matrix $A$ is uniformly elliptic and bounded, that is, there exist $\alpha$ and $\beta$ such that the following two conditions hold:
   (a) $\|A(x)v\| \le \beta\|v\|$ for all $v\in\mathbb{R}^2$ and for all $x\in\mathbb{H}^1$. Since $a_{i,j}$ for $i,j=1,2$ are $Y$-periodic, it is sufficient to hold for $x\in Y$.
   (b) For all $x\in\mathbb{H}^1$ or $x\in Y$ and $v\in\mathbb{R}^2$, $A$ satisfies $\langle A(x)v,v\rangle > \alpha\|v\|^2$.

For each $\varepsilon>0$, denote $A^\varepsilon(x)=A(\delta_{\frac{1}{\varepsilon}}(x))$. The map $x\to A^\varepsilon(x)$ can be realized as a moving frame with a section of the vector bundle of symmetric linear endomorphisms of the horizontal fibers. As an application of the unfolding operator on the Heisenberg group, we will consider the following homogenization problem: for $f\in L^2(\Omega)$, consider

$$\begin{cases} -\mathrm{div}_{\mathbb{H}}(A^\varepsilon\nabla_{\mathbb{H}}u_\varepsilon)+u_\varepsilon=f \text{ in } \Omega, \\ A^\varepsilon(x)\nabla_{\mathbb{H}}u_\varepsilon\cdot n_{\mathbb{H}}(x)=0 \text{ on } \partial\Omega. \end{cases} \tag{4.1}$$

Here $n_{\mathbb{H}}=C(x)\nu$, where $\nu$ is the Euclidean outward normal on $\partial\Omega$. More precisely, we are considering the following variational problem: find $u_\varepsilon\in H^1_{\mathbb{H}}(\Omega)$ such that

$$\int_\Omega A^\varepsilon\nabla_{\mathbb{H}}u_\varepsilon\cdot\nabla_{\mathbb{H}}\phi\,dx+\int_\Omega u_\varepsilon\phi\,dx=\int_\Omega f\phi\,dx \text{ for all } \phi\in H^1_{\mathbb{H}}(\Omega). \tag{4.2}$$

For every $\varepsilon>0$, The Lax–Milgram theorem guaranties the existence of the unique solution $u_\varepsilon$. By taking $u_\varepsilon$ as a test function on both side of (4.2), we get $\|u_\varepsilon\|_{H^1_{\mathbb{H}}(\Omega)}\le \frac{1}{\alpha}\|f\|_{L^2(\Omega)}$, where $\alpha$ is the elliptic constant. Our goal is to analyze the asymptotic behavior of the sequence of solution $u_\varepsilon$ as the periodic parameter $\varepsilon\to 0$. The present problem is not new, and it can be studied via two-scale convergence also. But our aim in this article is to introduce the unfolding operator, and through this standard example, we are exhibiting the easy way of studying the problem using the unfolding operator. Subsequently, we study nontrivial optimal control problems like optimal control problems with high contrasting diffusive coefficients.

The limiting behavior of the sequence of solutions $u_\varepsilon$ is summed up in the following theorem.

THEOREM 21 (two-scale limit theorem). *Let $\{u_\varepsilon\}$ be the sequence of solutions to* (4.1). *Then*

$$T^\varepsilon(u_\varepsilon)\to u \text{ strongly in } L^2(\Omega\times Y),$$
$$T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)\rightharpoonup\nabla_{\mathbb{H}}u+\nabla_{\mathbb{H},y}u_1 \text{ weakly in } (L^2(\Omega\times Y))^2,$$

*where $u\in H^1_{\mathbb{H}}(\Omega)$ is independent of $y$, and $(u,u_1)\in H^1_{\mathbb{H}}(\Omega)\times L^2(\Omega;H^1_{\#,\mathbb{H}}(Y)/\mathbb{R})$ satisfies the two-scale variational system*

$$\begin{aligned}\int_{\Omega\times Y} A(y)(\nabla_{\mathbb{H}}u(x)+\nabla_{\mathbb{H},y}u_1(x,y))\cdot(\nabla_{\mathbb{H}}\phi(x)+\nabla_{\mathbb{H},y}\phi_1(x,y))\,dxdy \\ +\int_{\Omega\times Y}u(x)\phi(x)\,dxdy=\int_{\Omega\times Y}f(x)\phi(x)\,dxdy\end{aligned} \tag{4.3}$$

*for all $(\phi,\phi_1)\in H^1_{\mathbb{H}}(\Omega)\times L^2(\Omega;H^1_{\#,\mathbb{H}}(Y)/\mathbb{R})$.*

*Proof.* To prove the above theorem, the periodic unfolding operator on the Heisenberg group will be used as the main tool. Since, we have the uniform bound on $\|u_\varepsilon\|_{H^1_{\mathbb{H}}(\Omega)}$, from Theorem 17, up to a subsequence, we have the existence of $(u,u_1)\in H^1_{\mathbb{H}}(\Omega)\times L^2(\Omega;H^1_{\#,\mathbb{H}}(Y)/\mathbb{R})$ such that

$$T^\varepsilon(u_\varepsilon)\to u \text{ strongly in } L^2(\Omega\times Y),$$
$$T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)\rightharpoonup \nabla_{\mathbb{H}}u+\nabla_{\mathbb{H},y}u_1 \text{ weakly in } L^2(\Omega\times Y).$$

The proof will be completed if we are able to show that $(u,u_1)$ satisfies the variational form (4.3). The oscillating test function will be used to prove that $(u,u_1)$ is the solution to the limit variational form. Let $\phi\in C^\infty(\bar\Omega)$ and $\phi_1\in C_c^\infty(\Omega;C^\infty_{\#,\mathbb{H}}(Y))$. Let $\phi_1^\varepsilon(x)=\varepsilon\phi_1(x,\delta_{\frac{1}{\varepsilon}}(x))$. Then, we have the following convergence:

$$T^\varepsilon(\phi)\to\phi \text{ strongly in } L^2(\Omega\times Y),$$
$$T^\varepsilon(\phi_1^\varepsilon)\to 0 \text{ strongly in } L^2(\Omega\times Y).$$

Now, by the periodicity of $\phi_1$ in $y$ and scale decomposition (3.1), we have $\phi_1^\varepsilon(x)=\varepsilon\phi_1(x,\delta_{\frac{1}{\varepsilon}}(x))=\varepsilon\phi_1(x,\{\delta_{\frac{1}{\varepsilon}}(x)\}_{\mathbb{H}})$. Now, apply $X_1$ on $\phi_1^\varepsilon$. Using the homogeneous property of the horizontal vector fields with respect to the dilation $\delta_\lambda$, we get

$$\begin{aligned}X_1(\phi_1^\varepsilon(x))&=\varepsilon\frac{\partial\phi_1}{\partial x_1}\left(x,\left\{\delta_{\frac{1}{\varepsilon}}(x)\right\}_{\mathbb{H}}\right)+\varepsilon 2x_2\frac{\partial\phi_1}{\partial x_3}\left(x,\left\{\delta_{\frac{1}{\varepsilon}}(x)\right\}_{\mathbb{H}}\right)\\&\quad+\frac{\partial\phi_1}{\partial y_1}\left(x,\left\{\delta_{\frac{1}{\varepsilon}}(x)\right\}_{\mathbb{H}}\right)+2\left\{\frac{x_2}{\varepsilon}\right\}_e\frac{\partial\phi_1}{\partial y_3}\left(x,\left\{\delta_{\frac{1}{\varepsilon}}(x)\right\}_{\mathbb{H}}\right)\\&=\varepsilon X_1\phi_1\left(x,\left\{\delta_{\frac{1}{\varepsilon}}(x)\right\}_{\mathbb{H}}\right)+Y_1\phi_1\left(x,\left\{\delta_{\frac{1}{\varepsilon}}(x)\right\}_{\mathbb{H}}\right).\end{aligned}$$

Similarly, we compute $X_2(\phi_1^\varepsilon(x))=\varepsilon X_2\phi_1\left(x,\left\{\delta_{\frac{1}{\varepsilon}}(x)\right\}_{\mathbb{H}}\right)+Y_2\phi_1\left(x,\left\{\delta_{\frac{1}{\varepsilon}}(x)\right\}_{\mathbb{H}}\right)$. Combining the above two equalities, we get the following relation:

$$\nabla_{\mathbb{H}}(\phi_1^\varepsilon(x))=\varepsilon\nabla_{\mathbb{H}}\phi_1\left(x,\delta_{\frac{1}{\varepsilon}}x\right)+\nabla_{\mathbb{H},y}\phi_1\left(x,\delta_{\frac{1}{\varepsilon}}x\right).$$

Now applying the unfolding operator on both sides and passing to the limit, we get

$$T^\varepsilon(\nabla_{\mathbb{H}}\phi_1^\varepsilon)\to\nabla_{\mathbb{H},y}\phi_1 \text{ strongly in } L^2(\Omega\times Y).$$

Since $\phi+\phi_1^\varepsilon\in H^1_{\mathbb{H}}(\Omega)$, we can use this as a test function in the weak formulation (4.2) to obtain

$$\int_\Omega A^\varepsilon\nabla_{\mathbb{H}}u_\varepsilon\cdot(\nabla_{\mathbb{H}}\phi+\nabla_{\mathbb{H}}\phi_1^\varepsilon)\,dx+\int_\Omega u_\varepsilon(\phi+\phi_1^\varepsilon)\,dx=\int_\Omega f(\phi+\phi_1^\varepsilon)dx.$$

Applying the unfolding operator on both sides of the variational form, we get

$$\begin{aligned}&\int_{\Omega\times Y}T^\varepsilon(A^\varepsilon\nabla_{\mathbb{H}}u_\varepsilon)(x,y)\cdot T^\varepsilon(\nabla_{\mathbb{H}}\phi+\nabla_{\mathbb{H}}\phi_1^\varepsilon)(x,y)\,dxdy\\&\quad+\int_{\Omega\times Y}T^\varepsilon(u_\varepsilon)(x,y)T^\varepsilon(\phi+\phi_1^\varepsilon)(x,y)\,dxdy=\int_{\Omega\times Y}T^\varepsilon(f)(x,y)T^\varepsilon(\phi+\phi_1^\varepsilon)(x,y)\,dxdy\end{aligned}$$

for all $\phi\in H^1_{\mathbb{H}}(\Omega)$. As $A$ is $Y$-periodic, it implies that $T^\varepsilon(A^\varepsilon)(x,y)=A(y)$. Using the convergence of $T^\varepsilon(u_\varepsilon)$, $T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)$, and $T^\varepsilon(\phi_1^\varepsilon)$, we can pass to the limit as $\varepsilon\to 0$ in the above integral equality to obtain

$$\int_{\Omega\times Y} A(y)(\nabla_{\mathbb{H}} u(x) + \nabla_{\mathbb{H},y} u_1(x,y)) \cdot (\nabla_{\mathbb{H}} \phi(x) + \nabla_{\mathbb{H},y}\phi_1(x,y))\, dxdy + \int_{\Omega\times Y} u(x)\phi(x)\, dxdy = \int_{\Omega\times Y} f(x)\phi(x)\, dxdy \tag{4.4}$$

for all $(\phi,\phi_1) \in C^\infty(\bar{\Omega}) \times C_c^\infty(\Omega; C^\infty_{\#,\mathbb{H}}(Y))$. By the density, we have that the above equality is true for all $(\phi,\phi_1)\in H^1_{\mathbb{H}}(\Omega)\times L^2(\Omega; H^1_{\#,\mathbb{H}}(Y)/\mathbb{R})$. In order to get the convergence of the full sequence it is sufficient to show that the limit variational form (4.3) admits a unique solution. Uniqueness will be proved if we establish that the following bilinear form $B : H^1_{\mathbb{H}}(\Omega) \times L^2(\Omega; H^1_{\#,\mathbb{H}}(Y)/\mathbb{R}) \times H^1_{\mathbb{H}}(\Omega) \times L^2(\Omega; H^1_{\#,\mathbb{H}}(Y)/\mathbb{R}) \to \mathbb{R}$, given by

$$B((u,u_1),(\phi,\phi_1)) = \int_{\Omega\times Y} A(y)(\nabla_{\mathbb{H}} u + \nabla_{\mathbb{H},y} u_1)\cdot(\nabla_{\mathbb{H}}\phi + \nabla_{\mathbb{H},y}\phi_1)\, dxdy + \int_{\Omega\times Y} u\phi\, dxdy$$

is elliptic. The ellipticity of $B$ follows from the ellipticity of $A$, that is,

$$B((\phi,\phi_1),(\phi,\phi_1)) > \frac{\alpha}{2}(\|\phi\|^2_{H^1(\Omega)} + \|\phi_1\|^2_{L^2(\Omega;H^1_{\#}(Y)/\mathbb{R})}).$$

This completes the proof of the theorem. □

**One-scale problem:** We can write the variational form (4.3) which is in two-scales in a more explicit way using the cell problem. More precisely, we can get the one-scale form (homogenized equation). In order to write the scale separated form, let us put $\phi = 0$ in (4.3) to get

$$\int_{\Omega\times Y} A(y)(\nabla_{\mathbb{H}} u(x) + \nabla_{\mathbb{H},y} u_1(x,y)) \cdot \nabla_{\mathbb{H},y}\phi_1(x,y)\, dxdy = 0. \tag{4.5}$$

Consider the following cell problem, for $i = 1, 2$, find $Z_i \in H^1_{\#,\mathbb{H}}(Y)/\mathbb{R}$ such that

$$\int_Y A(y)\nabla_{\mathbb{H},y} Z_i(y)\cdot\nabla_{\mathbb{H},y}\xi(y)dy = -\int_Y A(y)e_i\cdot\nabla_{\mathbb{H},y}\xi(y)dy \tag{4.6}$$

for all $\xi \in H_{\#,\mathbb{H}}(Y)/\mathbb{R}$. Here $e_i$ for $i = 1, 2$ denotes the standard basis for $\mathbb{R}^2$. Using $Z_i$, we can write $u_1(x,y) = \sum_{i=1}^2 Z_i(y)X_i u(x)$. Now put $\phi_1 = 0$ in the variational form (4.3) and substitute $u_1(x,y) = \sum_{i=1}^2 Z_i(y)X_i u(x)$ to get

$$\int_{\Omega\times Y} A(y)\left(\nabla_{\mathbb{H}} u(x) + \sum_{i=1}^2 \nabla_{\mathbb{H},y} Z_i X_i u\right)\cdot\nabla_{\mathbb{H}}\phi\, dxdy + |Y|\int_\Omega u\phi\, dx = |Y|\int_\Omega f\phi\, dx.$$

The equality above can be written as

$$\int_\Omega\left(\int_Y A(y)(I_{2\times 2} + \left[\nabla_{\mathbb{H},y}Z_1\ \nabla_{\mathbb{H},y}Z_2\right]\right)\nabla_{\mathbb{H}} u\cdot\nabla_{\mathbb{H}}\phi\, dx + |Y|\int_\Omega u\phi\, dx = |Y|\int_\Omega f\phi\, dx. \tag{4.7}$$

Denote the homogenized constant coefficient matrix

$$A_0 = \int_Y A(y)(I_{2\times 2} + \left[\nabla_{\mathbb{H},y}Z_1\ \nabla_{\mathbb{H},y}Z_2\right] dy) = \int_Y A(y)\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}Y_1Z_1 & Y_1Z_2\\ Y_2Z_1 & Y_2Z_2\end{bmatrix}\right) dy.$$

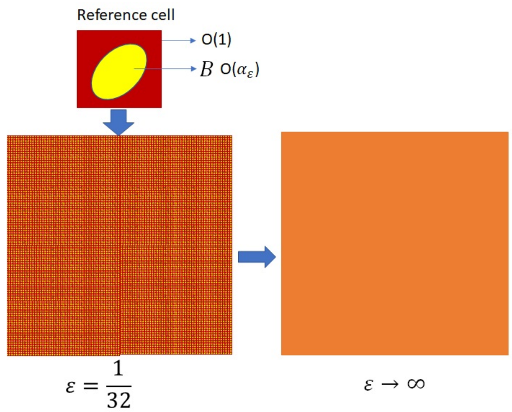


FIG. 2. *Composite material.*

Hence the variational form (4.7) reduces to

$$\int_\Omega A_0 \nabla_{\mathbb{H}} u \cdot \nabla_{\mathbb{H}} \phi \, dx + |Y| \int_\Omega u\phi \, dx = |Y| \int_\Omega f\phi \, dx. \tag{4.8}$$

The above variational form holds for all $\phi \in H^1_{\mathbb{H}}(\Omega)$. Hence the the varional form (4.8) corresponds to the following strong form:

$$\begin{cases} -\mathrm{div}_{\mathbb{H}}(A_0 \nabla_{\mathbb{H}} u) + |Y| u = & |Y| f \text{ in } \Omega, \\ A_0 \nabla_{\mathbb{H}} u \cdot n_{\mathbb{H}} = & 0 \text{ on } \partial\Omega. \end{cases} \tag{4.9}$$

This is the homogenized system corresponding to (4.1).

**4.1. Optimal control problem with strong contrasting diffusivity.** Now we will see homogenization of an optimal control problem having strong-contrasting diffusivity coefficients in the state equation or constrained PDE in a bounded domain $\Omega \subset \mathbb{H}^1$. First of all, we remark that we do not have a uniform bound on the solutions due to the lack of a uniform ellipticity constant in $\varepsilon$. We have to exploit the different bound obtained in the high contrasting and low contrasting regions. Further, the limit analysis depends on the applications of control whether it is on the conductive part or the nearly insulating part. Let us describe the setting of the problem. Recall that $Y = [0,2]^3$ and $E_\varepsilon = \{k \in \mathbb{Z}^3 : Y_k^\varepsilon \subset \Omega\}$. See Figure 2 in the Euclidean setup and see [24] for a study with hyperbolic problem.

Let $M, B \subset Y$ such that $Y = (\overline{B \cup M})^\circ$. We also assume that $M$ is simply connected with $C^{1,1}$-boundary in the usual sense. Now, for $k \in E_\varepsilon$, define $B_k^\varepsilon = \delta_\varepsilon(2k \cdot I)$, $M_k^\varepsilon = \delta_\varepsilon(2k \cdot M)$. We define $B_\varepsilon$ and $M_\varepsilon$ as $B_\varepsilon = \{\bigcup_{k\in E_\varepsilon} B_k^\varepsilon\} \cup \Lambda_\varepsilon$, $M_\varepsilon = \bigcup_{k \in E_\varepsilon} M_k^\varepsilon$.

**4.1.1. Control acting on $M_\varepsilon$.** For each $\varepsilon > 0$, let $L^2(M_\varepsilon)$ be the admissible control set. Let $A^\varepsilon(x)$, $\Omega_\varepsilon$ be as defined earlier. For $\theta_\varepsilon \in L^2(M_\varepsilon)$, consider the following $L^2$-cost functional, $J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2} \int_\Omega |u_\varepsilon - u_d|^2 \, dx + \frac{\rho}{2} \int_{M_\varepsilon} |\theta_\varepsilon|^2 \, dx$, where $\rho > 0$ is a regularization parameter and $u_\varepsilon$ satisfies the following constrained PDE,

$$\begin{cases} -\mathrm{div}_{\mathbb{H}}((\chi_{M_\varepsilon} + \varepsilon^2 \chi_{B_\varepsilon}) A^\varepsilon \nabla_{\mathbb{H}} u_\varepsilon) + u_\varepsilon = f + \chi_{M_\varepsilon} \theta_\varepsilon \text{ in } \Omega, \\ A^\varepsilon(x) \nabla_{\mathbb{H}} u_\varepsilon \cdot n_{\mathbb{H}} = 0 \text{ on } \partial\Omega \end{cases} \tag{4.10}$$

with $f \in L^2(\Omega)$. The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1_{\mathbb{H}}(\Omega) \times L^2(M_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon) : (u_\varepsilon, \theta_\varepsilon) \text{ satisfies (4.10)}\}. \tag{4.11}$$

Indeed the uniform bound is lost in the region $B_\varepsilon$. Note that $M_\varepsilon$ acts as the (highly) conductive region relative to $B_\varepsilon$ which is nearly an insulating region (see [24, 25, 28]). But, for each fixed $\varepsilon > 0$, $A^\varepsilon$ is uniformly elliptic and $\rho > 0$ implies that $J_\varepsilon$ is strictly convex. Hence, the classical method of calculus of variations ensures the existence and uniqueness of $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$. We aim to analyze the asymptotic behavior of the optimal solution $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ as $\varepsilon \to 0$. The following theorem gives the characterization of the optimal control at the $\varepsilon$ stage which is essential for our analysis.

THEOREM 22. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1_{\mathbb{H}}(\Omega) \times L^2_{\#,\mathbb{H}}(Y)$ be the optimal solution to the optimal control problem* (4.11). *Then, the optimal control $\bar{\theta}_\varepsilon$ can be written as*

$$T^\varepsilon(\bar{\theta}_\varepsilon) = \frac{1}{\rho} T^\varepsilon(\bar{v}_\varepsilon), \tag{4.12}$$

*where $\bar{v}_\varepsilon$ satisfies the following adjoint PDE:*

$$\begin{cases} -\mathrm{div}_{\mathbb{H}}((\chi_{M_\varepsilon} + \varepsilon^2 \chi_{B_\varepsilon}) A^\varepsilon \nabla_{\mathbb{H}} \bar{v}_\varepsilon) + \bar{v}_\varepsilon = (\bar{u_\varepsilon} - u_d) \ \text{in } \Omega, \\ A^\varepsilon \nabla_{\mathbb{H}} \bar{v}_\varepsilon \cdot n_{\mathbb{H}} = 0 \ \text{on } \partial\Omega. \end{cases} \tag{4.13}$$

*Conversely, suppose $(\hat{u}_\varepsilon, \hat{v}_\varepsilon, \hat{\theta}_\varepsilon)$ satisfies the following system,*

$$\begin{cases} -\mathrm{div}_{\mathbb{H}}((\chi_{M_\varepsilon} + \varepsilon^2 \chi_{B_\varepsilon}) A^\varepsilon \nabla_{\mathbb{H}} \hat{u}_\varepsilon) + \hat{u}_\varepsilon = f + \chi_{M_\varepsilon} \theta_\varepsilon \ \text{in } \Omega, \\ -\mathrm{div}_{\mathbb{H}}((\chi_{M_\varepsilon} + \varepsilon^2 \chi_{B_\varepsilon}) A^\varepsilon \nabla_{\mathbb{H}} \hat{v}_\varepsilon) + \hat{v}_\varepsilon = (\hat{u}_\varepsilon - u_d) \ \text{in } \Omega, \\ T^\varepsilon(\hat{\theta}_\varepsilon)(x,y) = -\chi_M(y) \frac{1}{\rho} T^\varepsilon(\hat{v}_\varepsilon)(x,y) \ \text{in } \Omega \end{cases} \tag{4.14}$$

*with the following boundary condition*

$$\begin{cases} A^\varepsilon(x) \nabla_{\mathbb{H}} \hat{u}_\varepsilon \cdot n_{\mathbb{H}} = 0 \ \text{ on } \partial\Omega, \\ A^\varepsilon \nabla_{\mathbb{H}} \hat{v}_\varepsilon \cdot n_{\mathbb{H}} = 0 \ \text{on } \partial\Omega. \end{cases} \tag{4.15}$$

*Then $(\hat{u}_\varepsilon, \hat{\theta}_\varepsilon)$ is the optimal solution to the optimal control problem.* (4.11)

*Proof.* Given $\theta_\varepsilon \in L^2(M_\varepsilon)$, denote $F_\varepsilon(\theta_\varepsilon) = J_\varepsilon(u_\varepsilon(\theta_\varepsilon), \theta_\varepsilon)$, where $u_\varepsilon(\theta_\varepsilon)$ is the solution to (4.10). Evaluating the limit of

$$\frac{1}{\lambda}(F_\varepsilon(\bar{\theta_\varepsilon} + \lambda\theta_\varepsilon) - F_\varepsilon(\bar{\theta}_\varepsilon))$$

as $\lambda \to 0$ and denoting the limit by $F'(\bar{\theta}_\varepsilon)\theta_\varepsilon$, we get

$$F'_\varepsilon(\bar{\theta}_\varepsilon)\theta_\varepsilon = \int_\Omega (\bar{u}_\varepsilon - u_d) w_{\theta_\varepsilon} \, dx + \rho \int_{M_\varepsilon} \bar{\theta}_\varepsilon \theta_\varepsilon \, dx.$$

Here $w_{\theta_\varepsilon}$ is the solution to the PDE

$$\begin{cases} -\mathrm{div}_{\mathbb{H}}\left((\chi_{M_\varepsilon} + \varepsilon^2 \chi_{B_\varepsilon}) A^\varepsilon \nabla_{\mathbb{H}} w_{\theta_\varepsilon}\right) + w_{\theta_\varepsilon} = \chi_{M_\varepsilon} \theta^\varepsilon \ \text{ in } \Omega, \\ A^\varepsilon \nabla_{\mathbb{H}} w_{\theta_\varepsilon} \cdot n_{\mathbb{H}} = 0 \ \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

As $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ is the optimal solution, we have $F'_\varepsilon(\bar{\theta}_\varepsilon)\theta_\varepsilon = 0$ for all $\theta_\varepsilon \in L^2(M_\varepsilon)$. Hence, we get

$$\int_\Omega (\bar{u}_\varepsilon - u_d) w_{\theta_\varepsilon}\, dx = -\rho \int_\Omega \chi_{M_\varepsilon} \bar{\theta}_\varepsilon \theta_\varepsilon\, dx. \tag{4.17}$$

Using $w_{\theta_\varepsilon}$ as a test function in (4.13) and $\bar{v}_\varepsilon$ in (4.16), we obtain

$$\int_\Omega (\bar{u_\varepsilon} - u_d) w_{\theta_\varepsilon}\, dx = \int_\Omega \chi_{M_\varepsilon} \bar{v}_\varepsilon \theta^\varepsilon\, dx. \tag{4.18}$$

Hence from (4.17) and (4.18), we have

$$\int_\Omega \chi_{M_\varepsilon} \bar{\theta}_\varepsilon \theta_\varepsilon\, dx = -\frac{1}{\rho} \int_\Omega \bar{v}_\varepsilon \chi_{M_\varepsilon} \theta_\varepsilon\, dx \qquad \text{for all } \theta_\varepsilon \in L^2(M_\varepsilon). \tag{4.19}$$

The above equality implies $\bar{\theta}_\varepsilon = -\frac{1}{\rho}\chi_{M_\varepsilon}\bar{v}_\varepsilon$. Hence the unfolding integral equality implies that

$$T^\varepsilon(\bar{\theta}_\varepsilon)(x,y) = -\frac{1}{\rho}\chi_M(y) T^\varepsilon(\bar{v_\varepsilon})(x,y).$$ ∎

To analyze the asymptotic behavior, we need to introduce the following Sobolev spaces:

$$H^1_{0,\mathbb{H}}(B) = \text{Closure of } C_c^\infty(B) \text{ in } H^1_{\mathbb{H}}(B) \text{ norm},$$
$$V^{\text{div}}_{\#,\mathbb{H}}(M) = \{\psi \in V^{\text{div}}_{\#,\mathbb{H}}(Y) : supp(\psi)|_Y \subset M\}.$$

As $0 \in L^2(M_\varepsilon)$, we have $J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \le J_\varepsilon(u^0_\varepsilon, 0)$, where $u^0_\varepsilon$ is the solution to the state equation (4.10), corresponding to $\theta_\varepsilon = 0$. This implies that $\|\bar{\theta}\|_{L^2(M_\varepsilon)} \le C$. The variational form of (4.10) is given by

$$\begin{aligned} &\int_\Omega (\chi_{M_\varepsilon} + \varepsilon^2 \chi_{B_\varepsilon}) A^\varepsilon \nabla_{\mathbb{H}} u_\varepsilon \cdot \nabla_{\mathbb{H}} \phi\, dx + \int_\Omega u_\varepsilon \phi\, dx \\ &\quad = \int_\Omega f\phi\, dx + \int_\Omega \chi_{M_\varepsilon} \theta_\varepsilon \phi\, dx \ \text{ for all } \phi \in H^1_{\mathbb{H}}(\Omega). \end{aligned} \tag{4.20}$$

By taking $\phi = \bar{u}_\varepsilon$ and $\theta_\varepsilon = \bar{\theta}_\varepsilon$ as a test function in the variational form, we have

$$\begin{aligned} &\int_\Omega (\chi_{M_\varepsilon} + \varepsilon^2 \chi_{B_\varepsilon}) A^\varepsilon \nabla_{\mathbb{H}} \bar{u}_\varepsilon \cdot \nabla_{\mathbb{H}} \bar{u}_\varepsilon\, dx + \int_\Omega \bar{u}_\varepsilon \bar{u}_\varepsilon\, dx \\ &\quad = \int_\Omega f\bar{u}_\varepsilon\, dx + \int_\Omega \chi_{M_\varepsilon} \bar{\theta}_\varepsilon \bar{u}_\varepsilon\, dx. \end{aligned} \tag{4.21}$$

By applying Hölder's inequality, boundedness of $\|\bar{\theta}_\varepsilon\|_{L^2(M_\varepsilon)}$, ellipticity of $A$, we infer the following:

$$\begin{aligned} &\|\bar{u}_\varepsilon\|_{L^2(\Omega)} + \|\chi_{M_\varepsilon} \nabla_{\mathbb{H}} \bar{u}_\varepsilon\|_{L^2(\Omega)} + \|\varepsilon \chi_{B_\varepsilon} \nabla_{\mathbb{H}} \bar{u}_\varepsilon\|_{L^2(\Omega)} \\ &\quad \le C(\|f\|_{L^2(\Omega)} + \|\chi_{M_\varepsilon} \bar{\theta}_\varepsilon\|_{L^2(\Omega)}) \le C. \end{aligned} \tag{4.22}$$

In a similar fashion, we also have the following bound on the adjoint state:

$$\begin{aligned} &\|\bar{v}_\varepsilon\|_{L^2(\Omega)} + \|\chi_{M_\varepsilon} \nabla_{\mathbb{H}} \bar{v}_\varepsilon\|_{L^2(\Omega)} + \|\varepsilon \chi_{B_\varepsilon} \nabla_{\mathbb{H}} \bar{v}_\varepsilon\|_{L^2(\Omega)} \\ &\quad \le C(\|\bar{u}_\varepsilon\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}) \le C. \end{aligned} \tag{4.23}$$

Note that we have the uniform bound of $\nabla_{\mathbb{H}} \bar{u}_\varepsilon$ in $L^2(M_\varepsilon)$, but the bound of $\nabla_{\mathbb{H}} \bar{u}_\varepsilon$ is of order $\varepsilon^{-1}$. In other words we do not have the uniform bound of $\bar{u}_\varepsilon$ in $H^1_{\mathbb{H}}(\Omega)$. Hence we cannot derive convergence directly. But the unfolding operator introduced earlier is helpful at this stage. The above bounds lead to the following convergence theorem.

THEOREM 23. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem* (4.11) *and $\bar{v}_\varepsilon$ be the adjoint state that satisfies* (4.13)*, then there exist $u, v \in L^2(\Omega), U_1, V_1 \in L^2(\Omega, H^1_{0,\mathbb{H}}(B))$, and $u_1, v_1 \in L^2(\Omega; H^1_{\#,\mathbb{H}}(M)/\mathbb{R})$ such that*

(i) $T^\varepsilon(\bar{u}_\varepsilon) \rightharpoonup u + \chi_B U_1$, $T^\varepsilon(\bar{v}_\varepsilon) \rightharpoonup v + \chi_B V_1$, *weakly in* $L^2(\Omega \times Y)$,

(ii) $T^\varepsilon(\chi_{M_\varepsilon} \nabla_{\mathbb{H}} \bar{u}_\varepsilon) \rightharpoonup \chi_M(\nabla_{\mathbb{H}} u + \nabla_{\mathbb{H},y} u_1)$, *weakly in* $(L^2(\Omega \times Y))^2$,
$T^\varepsilon(\chi_{M_\varepsilon} \nabla_{\mathbb{H}} \bar{v}_\varepsilon) \rightharpoonup \chi_M(\nabla_{\mathbb{H}} v + \nabla_{\mathbb{H},y} v_1)$, *weakly in* $(L^2(\Omega \times Y))^2$,

(iii) $T^\varepsilon(\varepsilon \chi_{B_\varepsilon} \nabla_{\mathbb{H}} \bar{u}_\varepsilon) \rightharpoonup \chi_B \nabla_{\mathbb{H},y} U_1$ *weakly in* $(L^2(\Omega \times Y))^2$,
$T^\varepsilon(\varepsilon \chi_{B_\varepsilon} \nabla_{\mathbb{H}} \bar{\bar{v}}_\varepsilon) \rightharpoonup \chi_B \nabla_{\mathbb{H},y} V_1$ weakly in $(L^2(\Omega \times Y))^2$,

(iv) $T^\varepsilon(\bar{\theta}_\varepsilon) \rightharpoonup -\chi_M \dfrac{1}{\rho} v$ *weakly in* $L^2(\Omega \times Y)$.

*The limits $(u, v, u_1, v_1, U_1, V_1)$ satisfy the following variational forms,*

$$
(4.24)\quad \begin{cases}
\int_{\Omega\times M} A(y)(\nabla_{\mathbb{H}} u + \nabla_{\mathbb{H},y} u_1)\cdot(\nabla_{\mathbb{H}}\phi(x) + \nabla_{\mathbb{H},y}\phi_1)\,dxdy \\
\quad + \int_{\Omega\times B} A(y)\nabla_{\mathbb{H},y} U_1 \nabla_{\mathbb{H},y} w\,dxdy + \int_{\Omega\times Y}(u + \chi_B U_1)(\phi + \chi_B w)\,dxdy \\
= \int_{\Omega\times Y}\left(f(x) - \chi_M \frac{1}{\rho} v\right)(\phi + \chi_B)\,dxdy, \\
\int_{\Omega\times M} A(y)(\nabla_{\mathbb{H}} v + \nabla_{\mathbb{H},y} v_1)\cdot(\nabla_{\mathbb{H}}\phi + \nabla_{\mathbb{H},y}\phi_1)\,dxdy \\
\quad + \int_{\Omega\times B} A(y)\nabla_{\mathbb{H},y} V_1 \nabla_{\mathbb{H},y} w\,dxdy + \int_{\Omega\times Y}(v + \chi_B V_1)(\phi + \chi_B w)\,dxdy \\
= \int_{\Omega\times Y}(u + \chi_B U_1 - u_d)(\phi + \chi_B w)\,dxdy
\end{cases}
$$

*for all $(\phi, \phi_1, w) \in H^1_{\mathbb{H}}(\Omega) \times L^2(\Omega; H^1_{\#,\mathbb{H}}(Y)/\mathbb{R}) \times L^2(\Omega; H^1_{0,\mathbb{H}}(B))$.*

*Proof.* *Step* 1. (4.22) and (4.23), and the integral equality property of the unfolding operator imply the following uniform bound on the unfolded sequence:

$$
\begin{aligned}
&\|T^\varepsilon(\bar{u}_\varepsilon)\|_{L^2(\Omega\times Y)}, \quad \|T^\varepsilon(\chi_{M_\varepsilon}\nabla_{\mathbb{H}}\bar{u}_\varepsilon)\|_{L^2(\Omega\times Y)}, \quad \|T^\varepsilon(\varepsilon\chi_{B_\varepsilon}\nabla_{\mathbb{H}}\bar{u}_\varepsilon)\|_{L^2(\Omega\times Y)} \le C, \\
&\|T^\varepsilon(\bar{v}_\varepsilon)\|_{L^2(\Omega\times Y)}, \quad \|T^\varepsilon(\chi_{M_\varepsilon}\nabla_{\mathbb{H}}\bar{v}_\varepsilon)\|_{L^2(\Omega\times Y)}, \quad \|T^\varepsilon(\varepsilon\chi_{B_\varepsilon}\nabla_{\mathbb{H}}\bar{v}_\varepsilon)\|_{L^2(\Omega\times Y)} \le C.
\end{aligned}
$$

Hence up to a subsequence, there exist $U, V \in L^2(\Omega\times Y)$, $\xi_1, \xi_2, K_1, K_1 \in (L^2(\Omega\times Y))^2$ such that

(i) $T^\varepsilon(\bar{u}_\varepsilon) \rightharpoonup U$, $T^\varepsilon(\bar{v}_\varepsilon) \rightharpoonup V$, weakly in $L^2(\Omega \times Y)$,
(ii) $T^\varepsilon(\chi_{M_\varepsilon}\nabla_{\mathbb{H}}\bar{u}_\varepsilon) \rightharpoonup \xi_1$, $T^\varepsilon(\varepsilon\chi_{B_\varepsilon}\nabla_{\mathbb{H}}\bar{u}_\varepsilon) \rightharpoonup \xi_2$, weakly in $(L^2(\Omega \times Y))^2$,
(ii) $T^\varepsilon(\chi_{M_\varepsilon}\nabla_{\mathbb{H}}\bar{v}_\varepsilon) \rightharpoonup K_1$, $T^\varepsilon(\varepsilon\chi_{B_\varepsilon}\nabla_{\mathbb{H}}\bar{v}_\varepsilon) \rightharpoonup K_2$, weakly in $(L^2(\Omega \times Y))^2$.

In the rest of the proof, we will identify $U, V, \xi_1, \xi_2, K_1$, and $K_2$.

*Step* 2 (identifying $U$ and $V$). We will show that $U$ can be decomposed as $U(x,y) = u(x) + U_1(x,y)$, where $u \in L^2(\Omega), U_1 \in L^2(\Omega \times Y)$, and $U_1(x,\cdot)|_M = 0$. To see this, for $\phi \in (C_c^\infty(\Omega; C^\infty_{\#,\mathbb{H}}(Y)))^2$ with $\mathrm{supp}(\phi(x,\cdot)|_Y) \subset M$, we have

$$
(4.25)\qquad \int_{\Omega\times M} \nabla_{\mathbb{H},y}(T^\varepsilon(\bar{u}_\varepsilon))(x,y)\phi(x,y)\,dxdy = \int_{\Omega\times M} \varepsilon T^\varepsilon(\nabla_{\mathbb{H}}(\bar{u}_\varepsilon))\phi(x,y)\,dxdy.
$$

On the other hand, we have

$$
(4.26)\qquad \int_{\Omega\times M} \nabla_{\mathbb{H},y}(T^\varepsilon(\bar{u}_\varepsilon))(x,y)\phi(x,y)\,dxdy = -\int_{\Omega\times M} \varepsilon T^\varepsilon(\bar{u}_\varepsilon)\mathrm{div}_{\mathbb{H},y}\phi(x,y)\,dxdy.
$$

By letting $\varepsilon \to 0$ in (4.25) and (4.26), we obtain, $\int_{\Omega\times Y} U(x,y)\mathrm{div}_{\mathbb{H},y}\phi(x,y)\,dxdy = 0$. Since $\phi$ is chosen arbitrarily, we infer that $U(x,\cdot)$ is independent of $y$ on $M$. Let

$U(x,y)|_{\Omega\times M}=u(x)$. Define $U_1=U-u$. Here we can see $u\in L^2(\Omega)$, $U_1\in L^2(\Omega\times Y)$, and $U=u+U_1$.

Following the same path, we can show the existence of $v\in L^2(\Omega)$ and $V_1\in L^2(\Omega\times Y)$ with $V_1(x,\cdot)|_M=0$ such that $V=v+V_1$. Hence from the characterization of optimal control, we have $T^\varepsilon(\bar\theta_\varepsilon)=-\dfrac{1}{\rho}\chi_M(y)T^\varepsilon(\bar v_\varepsilon)\rightharpoonup -\chi_M(y)\dfrac{1}{\rho}v$, weakly in $L^2(\Omega\times Y)$.

*Step* 3 (identifying $\xi_1$ and $K_1$). To identify $\xi_1$, we will use test functions of the form $\psi_\varepsilon(x)=\psi\left(x,\delta_{\frac{1}{\varepsilon}}(x)\right)$, where $\psi\in(C_c^\infty(\Omega,C_c^\infty(M)))^2$ with $\mathrm{div}_{\mathbb{H},y}\psi=0$ which is extended $Y$-periodically. Consider

$$\begin{aligned}&\int_\Omega \nabla_{\mathbb{H}}u_\varepsilon\chi_{M_\varepsilon}\psi\left(x,\delta_{\frac{1}{\varepsilon}}(x)\right)dx\\&\quad=\frac{1}{|Y|}\int_{\Omega\times M}T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)(x,y)\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right),y\right)dxdy.\end{aligned}$$

Recall that $T^\varepsilon(\nabla_{\mathbb{H}}u_\varepsilon)\rightharpoonup\xi_1$, weakly in $(L^2(\Omega\times Y))^2$. Now, using integration by parts and the gradient relation between $\nabla_{\mathbb{H}}$ and $\nabla_{\mathbb{H},y}$, we get

$$\begin{aligned}&\int_{\Omega\times M}T^\varepsilon(\nabla_{\mathbb{H}}\bar u_\varepsilon)(x,y)\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right),y\right)dxdy\\&\quad=\int_{\Omega\times M}\frac{1}{\varepsilon}\nabla_{\mathbb{H},y}T^\varepsilon(\bar u_\varepsilon)(x,y)\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right),y\right)dxdy\\&\quad=-\int_{\Omega\times M}T^\varepsilon(\bar u_\varepsilon)(x,y)\left(\mathrm{div}_{\mathbb{H}}\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right),y\right)\right)dxdy\\&\quad=-\int_{\Omega\times M}T^\varepsilon(\bar u_\varepsilon)(x,y)T^\varepsilon(\mathrm{div}_{\mathbb{H}}\psi)(x,y)\,dxdy.\end{aligned}$$

By passing to the limit on the both sides of the above equality, we obtain

$$\int_{\Omega\times M}\xi_1(x,y)\psi(x,y)\,dxdy=-\int_{\Omega\times M}u(x)\mathrm{div}_{\mathbb{H}}\psi(x,y)\,dxdy. \tag{4.27}$$

The above equality holds for arbitrary $\psi\in(C_c^\infty(\Omega,C_c^\infty(M)))^2$ with $\mathrm{div}_{\mathbb{H},y}\psi=0$. Since we have assumed $M$ has a $C^{1,1}$-boundary, from the proof of Lemma 3.3 in [13], we infer that $u\in H^1_{0,\mathbb{H}}(\Omega)$. Now we can perform integration by parts on the right-hand side of (4.27) to obtain $\int_{\Omega\times M}(\xi_1-\nabla_{\mathbb{H}}u(x))\psi(x,y)\,dxdy=0$. We have $\xi_1,\nabla_{\mathbb{H}}u(x)\in(L^2(\Omega\times M))^2$. Hence $\xi_1(x,y)-\nabla_{\mathbb{H}}u(x)\in L^2(\Omega;(V^{\mathrm{div}}_{\#,\mathbb{H}}(M))^*)$. Also, since $(C_c^\infty(\Omega,C_c^\infty(M)))^2$ is dense in $L^2(\Omega;(V^{\mathrm{div}}_{\#,\mathbb{H}}(M)))$, this implies that $\int_{\Omega\times M}(\xi_1-\nabla_{\mathbb{H}}u(x))\psi(x,y)dxdy=0$ for all $\psi\in L^2(\Omega;(V^{\mathrm{div}}_{\#,\mathbb{H}}(M)))$ with $\mathrm{div}_{\mathbb{H}}\psi=0$.
Hence, $\xi_1-\nabla_{\mathbb{H}}u$ is perpendicular to the divergence free vector field. We get from Theorem 5 that there exists a unique $u_1\in L^2(\Omega,L^2_{\#,\mathbb{H}}(M)/\mathbb{R})$ such that $\xi_1-\nabla_{\mathbb{H}}u=\nabla_{\mathbb{H},y}u_1$. Since, $\xi_1$ and $\nabla_{\mathbb{H}}u$ are in $(L^2(\Omega\times Y))^2$, we see that $u_1\in L^2(\Omega;H^1_{\#,\mathbb{H}}(M)/\mathbb{R})$. Hence, we have the second convergence (ii). In a similar way, we can establish the existence of $v_1\in L^2(\Omega;H^1_{\#,\mathbb{H}}(M)/\mathbb{R})$ such that $T^\varepsilon(\chi_{M_\varepsilon}\nabla\bar v_\varepsilon)\rightharpoonup K_1=\chi_M(\nabla_{\mathbb{H}}v+\nabla_{\mathbb{H},y}v_1)$, weakly in $(L^2(\Omega\times Y))^2$.

*Step* 4 (identifying $\xi_2$ and $K_2$). To identify $K_2$, we use oscillating test functions. Let $\psi\in(C_c^\infty(\Omega;C_c^\infty(B))^2$, by extending $Y$-periodically, define $\psi_\varepsilon(x)=\psi(x,\delta_{\frac{1}{\varepsilon}}(x))$, and consider

$$
\begin{aligned}
\int_\Omega \varepsilon \chi_{B_\varepsilon} \nabla_{\mathbb{H}} \bar{u}_\varepsilon \psi^\varepsilon &= \frac{1}{|Y|}\int_{\Omega\times Y} \varepsilon T^\varepsilon(\chi_{B_\varepsilon}\nabla_{\mathbb{H}} u_\varepsilon)(x,y) T^\varepsilon(\psi_\varepsilon)(x,y)\,dxdy \\
&= \frac{1}{|Y|}\int_{\Omega\times B} \chi_B(y) T^\varepsilon(\nabla_{\mathbb{H}} u_\varepsilon)(x,y)\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right), y\right)dxdy \\
&= \frac{1}{|Y|}\int_{\Omega\times B} \frac{\varepsilon}{\varepsilon}\nabla_{\mathbb{H},y} T^\varepsilon(u_\varepsilon)(x,y)\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right), y\right)dxdy \\
&= -\frac{1}{|Y|}\int_{\Omega\times B} T^\varepsilon(u_\varepsilon)(x,y)\left(\varepsilon \mathrm{div}_{\mathbb{H}}\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right), y\right)\right. \\
&\quad \left. + \mathrm{div}_{\mathbb{H}}\psi\left(\delta_\varepsilon\left(2\left[\delta_{\frac{1}{\varepsilon}}(x)\right]_{\mathbb{H}}\cdot y\right), y\right)\right) dxdy.
\end{aligned} \tag{4.28}
$$

By letting $\varepsilon \to 0$ in (4.28), we obtain

$$
\int_{\Omega\times B} \xi_2(x,y)\psi(x,y) = -\int_{\Omega\times B}(u(x)+U_1(x,y))\mathrm{div}_{\mathbb{H},y}\psi(x,y).
$$

Since $\psi \in C_c^\infty(\Omega; C_c^\infty(B))$ is chosen arbitrarily, we infer that $\xi_2 = \nabla_{\mathbb{H},y}U_1$ a.e. in $\Omega\times B$. Also $\xi_2$ is supported in $\Omega\times B$, hence $\xi_2 = \nabla_{\mathbb{H},y}U_1$ a.e. $\Omega\times B$. Similarly, we can show that $K_2 = \nabla_{\mathbb{H},y}V_1$. This also proves that $U_1, V_1 \in L^2(\Omega; H^1_{0,\mathbb{H}}(B))$.

*Step* 5. Let $\psi^\varepsilon(x) = \phi(x) + \varepsilon\phi_1(x, \delta_{\frac{1}{\varepsilon}}(x)) + w(x, \delta_{\frac{1}{\varepsilon}}(x))$, where $(\phi,\phi_1) \in H^1_{\mathbb{H}}(\Omega)\times L^2(\Omega; H^1_{\#,\mathbb{H}}(M)/\mathbb{R})$ and $w \in L^2(\Omega; H^1_{0,\mathbb{H}}(B))$. Using $\psi_\varepsilon$ as a test function in the variational forms (4.10) and (4.13) and passing to the limit $\varepsilon\to 0$ by using the convergences we have obtained in the previous steps, we arrived at the limit variational forms (4.24).

*Step* 6 (convergence of the whole sequence). The variational form (4.24) is the optimality system corresponding to the following optimal control problem: find $(\bar{u}, \bar{U}_1, \bar{\theta}) \in H^1_{\mathbb{H}}(\Omega)\times L^2(\Omega; H^1_{0,\mathbb{H}}(B))\times L^2(\Omega)$, such that $J(\bar{u}, \bar{U}_1, \bar{\theta}) = \inf J(u, U_1, \theta)$, where $(u, U_1, \theta)$ satisfies

$$
\begin{aligned}
&\int_{\Omega\times M} A(y)(\nabla_{\mathbb{H}} u + \nabla_{\mathbb{H},y}u_1)\cdot(\nabla_{\mathbb{H}}\phi + \nabla_{\mathbb{H},y}\phi_1)\,dxdy \\
&\quad + \int_{\Omega\times Y}(u+\chi_B U_1)(\phi(x)+\chi_B w)\,dxdy + \int_{\Omega\times B} A(y)\nabla_{\mathbb{H},y}U_1\nabla_{\mathbb{H},y}w\,dxdy \\
&= \int_{\Omega\times Y}(f+\chi_M\theta)(\phi+\chi_B w)\,dxdy
\end{aligned} \tag{4.29}
$$

for all $(\phi,\phi_1,w)\in H^1_{\mathbb{H}}(\Omega)\times L^2(\Omega; H^1_{\#,\mathbb{H}}(M)/\mathbb{R})\times L^2(\Omega; H^1_{0,\mathbb{H}}(B))$, and the cost functional $J$ defined as

$$
J(u, U_1, \theta) = \frac{1}{2}\int_{\Omega\times Y}(u+\chi_B U_1 - u_d)^2\,dxdy + \frac{\rho}{2}\int_\Omega |M||\theta|^2\,dx. \tag{4.30}
$$

As the cost functional is strictly convex, it ensures the uniqueness of the solution to the optimality system. Hence all the subsequential limits are nothing but the limit of the full sequence. ∎

**Scales separation.** In the above equation, we have arrived at the limit optimal control problem in the unfolded space with a combined macro-micro scale. Here in this subsection, we want to see the limit optimal control in the scale separated form, in the macro scale only. Let us start with the state variational form (4.29). In variational form (4.29) putting $\phi, w = 0$, we get

$$\int_{\Omega\times M} A(y)(\nabla_{\mathbb{H}} u(x) + \nabla_{\mathbb{H},y} u_1(x,y)) \cdot \nabla_{\mathbb{H},y}\phi_1(x,y)\, dxdy = 0. \tag{4.31}$$

Let us introduce the following cell problem: for $i = 1, 2$, find $Z_i \in H^1_{\#,\mathbb{H}}(Y)/\mathbb{R}$ such that $\int_M A(y)\nabla_{\mathbb{H},y} Z_i(y) \cdot \nabla_{\mathbb{H},y}\xi(y)dy = -\int_M A(y)e_i \cdot \nabla_{\mathbb{H},y}\xi(y)dy$ for all $\xi \in H^1_{\#,\mathbb{H}}(M)/\mathbb{R}$. Here $e_i$ for $i = 1, 2$ denotes the standard basis for $\mathbb{R}^2$. Using $Z_i$, we can write $u_1(x,y) = \sum_{i=1}^2 Z_i(y)X_i u(x)$. Now put $\phi, \phi_1 = 0$ in the variational form (4.29) to get

$$\int_{\Omega\times B} A(y)\nabla_{\mathbb{H},y} U_1 \nabla_{\mathbb{H},y} w\, dxdy + U_1 w\, dxdy = \int_{\Omega\times Y} (f-u)w\, dxdy. \tag{4.32}$$

As $f, u$ are independent of $y$, we can write $U_1(x,y) = (f(x) - u(x))\eta(y)$, where $\eta$ satisfies the following variational cell problem:

$$\int_B (A(y)\nabla_{\mathbb{H},y}\eta(y)\nabla_{\mathbb{H},y} w(y) + \eta(y)w(y))\, dy = \int_B w(y)\, dy \text{ for all } w \in H^1_{0,\mathbb{H}}(B). \tag{4.33}$$

Now putting $\phi_1, w = 0$ in (4.29), and substituting the explicit form of $u_1$ and $U_1$, we obtain

$$\begin{aligned}\int_\Omega A_0(x)\nabla_{\mathbb{H}} u(x)\nabla_{\mathbb{H}}\phi(x)\, dx + \int_\Omega \left(|Y| - \int_B \eta(\tau)d\tau\right) u(x)\phi(x)\, dx \\ = \int_\Omega \left(|Y| - \int_B \eta(\tau)d\tau\right) f\phi\, dx + \int_\Omega |M|\theta\phi\, dx,\end{aligned} \tag{4.34}$$

where $A_0$ is the homogenized limit given by

$$A_0 = \int_M A(y) \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} Y_1 Z_1 & Y_1 Z_2 \\ Y_2 Z_1 & Y_2 Z_2 \end{bmatrix} \right) dy.$$

The cost functional reduces to

$$\bar{J}(u,\theta) = \frac{1}{2}\int_\Omega\int_Y |(1-\eta)u + f\eta - u_d|^2\, dxdy + \frac{\beta}{2}\int_\Omega |M||\theta|^2\, dx.$$

The optimal optimal control is given by find $(\bar{u}, \bar{\theta})) \in H^1_{\mathbb{H}}(\Omega) \times L^2(\Omega)$ such that

$$J(\bar{u}, \bar{\theta}) = \inf\{J(u,\theta) : (u,\theta) \text{ satisfies } (4.34)\}.$$

One can indeed write a one-scale homogenization for the macrovariable but we omit it.

**4.1.2. Control acting on $B_\varepsilon$.** Here we will see the effect of controls when they are applied in the low diffusive part. We begin by recalling that a two-scale homogenization theorem can be proved in a similar fashion as earlier, but the controls acting on $B_\varepsilon$ behaves differently and the two-scale system is not completely decomposable to provide a one-scale homogenization system. We will see the difficulty below. For $\varepsilon > 0$, let the admissible control set be $L^2(B_\varepsilon)$. The $L^2$-cost functional $J_\varepsilon$ is given by $J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2}\int_\Omega |u_\varepsilon - u_d|^2\, dx + \frac{\rho}{2}\int_{B_\varepsilon} |\theta_\varepsilon|^2\, dx$, where $\rho > 0$ is a regularization parameter and $u_\varepsilon$ satisfies the following constrained PDE:

$$\begin{cases} -\mathrm{div}_{\mathbb{H}}((\chi_{M_\varepsilon} + \varepsilon^2\chi_{B_\varepsilon})A^\varepsilon\nabla_{\mathbb{H}} u_\varepsilon) + u_\varepsilon = f + \chi_{B_\varepsilon}\theta_\varepsilon \text{ in } \Omega, \\ A^\varepsilon(x)\nabla_{\mathbb{H}} u_\varepsilon \cdot n_{\mathbb{H}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.35}$$

The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1_{\mathbb{H}}(\Omega) \times L^2(B_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon) : (u_\varepsilon, \theta_\varepsilon) \text{ satisfies } (4.35)\}. \tag{4.36}$$

Here also, we have a similar type of characterization theorem, as in Theorem 22.

THEOREM 24. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1_{\mathbb{H}}(\Omega) \times L^2_{\#,\mathbb{H}}(Y)$ be the optimal solution to the optimal control problem* (4.36). *Then, the optimal control $\bar{\theta}_\varepsilon$ can be written as $T^\varepsilon(\bar{\theta}_\varepsilon) = -\chi_B(y)\frac{1}{\rho}T^\varepsilon(\bar{v}_\varepsilon)$, where $\bar{v}_\varepsilon$ satisfies the following adjoint PDE:*

$$\begin{cases} -\mathrm{div}_{\mathbb{H}}((\chi_{M_\varepsilon} + \varepsilon^2\chi_{B_\varepsilon})A^\varepsilon \nabla_{\mathbb{H}} \bar{v}_\varepsilon) + \bar{v}_\varepsilon = (\bar{u_\varepsilon} - u_d) \text{ in } \Omega, \\ A^\varepsilon \nabla_{\mathbb{H}} \bar{v}_\varepsilon \cdot n_{\mathbb{H}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.37}$$

*Conversely, suppose $(\hat{u}_\varepsilon, \hat{v}_\varepsilon, \hat{\theta}_\varepsilon)$ satisfies the following optimality system:*

$$\begin{cases} -\mathrm{div}_{\mathbb{H}}((\chi_{M_\varepsilon} + \varepsilon^2\chi_{B_\varepsilon})A^\varepsilon \nabla_{\mathbb{H}} \hat{u}_\varepsilon) + \hat{u}_\varepsilon = f + \chi_{B_\varepsilon}\theta_\varepsilon \text{ in } \Omega, \\ -\mathrm{div}_{\mathbb{H}}((\chi_{M_\varepsilon} + \varepsilon^2\chi_{B_\varepsilon})A^\varepsilon \nabla_{\mathbb{H}} \hat{v}_\varepsilon) + \hat{v}_\varepsilon = (\hat{u}_\varepsilon - u_d) \text{ in } \Omega, \\ T^\varepsilon(\hat{\theta}_\varepsilon)(x,y) = -\chi_B(y)\frac{1}{\rho}T^\varepsilon(\hat{v}_\varepsilon)(x,y) \text{ in } \Omega, \\ A^\varepsilon(x)\nabla_{\mathbb{H}}\hat{u}_\varepsilon \cdot n_{\mathbb{H}} = 0, \quad A^\varepsilon \nabla_{\mathbb{H}}\hat{v}_\varepsilon \cdot n_{\mathbb{H}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.38}$$

*Then $(\hat{u}_\varepsilon, \hat{\theta}_\varepsilon)$ is the optimal solution to the optimal control problem* (4.36).

Following the same path as in the previous case, we obtain the following convergence result and homogenized two-scale limit optimality system similarly to Theorem 23. We will not present the details as the analysis is similar.

THEOREM 25. *Let $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon)$ be the optimal solution to the optimal control problem* (4.11) *and $\bar{v}_\varepsilon$ be the adjoint state satisfying* (4.37), *then there exist $u, v \in L^2(\Omega)$, $U_1, V_1 \in L^2(\Omega, H^1_{0,\mathbb{H}}(B))$, and $u_1, v_1 \in L^2(\Omega; H^1_{\#,\mathbb{H}}(M))$ such that*

(i) $T^\varepsilon(\bar{u}_\varepsilon) \rightharpoonup u + \chi_B U_1$, $T^\varepsilon(\bar{v}_\varepsilon) \rightharpoonup v + \chi_B V_1$, *weakly in* $L^2(\Omega \times Y)$,

(ii) $T^\varepsilon(\chi_{M_\varepsilon}\nabla_{\mathbb{H}}\bar{u}_\varepsilon) \rightharpoonup \chi_M(y)(\nabla_{\mathbb{H}}u + \nabla_{\mathbb{H},y}u_1)$, *weakly in* $(L^2(\Omega \times Y))^2$, $T^\varepsilon(\chi_{M_\varepsilon}\nabla_{\mathbb{H}}\bar{v}_\varepsilon) \rightharpoonup \chi_M(y)(\nabla_{\mathbb{H}}v + \nabla_{\mathbb{H},y}v_1)$, *weakly in* $(L^2(\Omega \times Y))^2$,

(iii) $T^\varepsilon(\varepsilon\chi_{B_\varepsilon}\nabla_{\mathbb{H}}\bar{u}_\varepsilon) \rightharpoonup \chi_B \nabla_{\mathbb{H},y}U_1$ *weakly in* $(L^2(\Omega \times Y))^2$,

(iv) $T^\varepsilon(\bar{\theta}_\varepsilon) \rightharpoonup -\chi_B\frac{1}{\rho}(v + V_1)$, *weakly in* $L^2(\Omega \times Y)$. *The limits* $(u, v, u_1, v_1, U_1, V_1)$ *satisfy the following variational forms,*

$$\begin{cases} \displaystyle\int_{\Omega\times M} A(y)(\nabla_{\mathbb{H}}u + \nabla_{\mathbb{H},y}u_1)\cdot(\nabla_{\mathbb{H}}\phi + \nabla_{\mathbb{H},y}\phi_1)\,dxdy \\ \qquad + \int_{\Omega\times B} A(y)\nabla_{\mathbb{H},y}U_1\nabla_{\mathbb{H},y}w\,dxdy + \int_{\Omega\times Y}(u + \chi_B U_1)(\phi + \chi_B w)\,dxdy \\ \displaystyle = \int_{\Omega\times Y}\left(f - \chi_B\frac{1}{\rho}(v + \chi_B V_1)\right)(\phi + \chi_B w)\,dxdy, \\ \\ \displaystyle\int_{\Omega\times M} A(y)(\nabla_{\mathbb{H}}v + \nabla_{\mathbb{H},y}v_1)\cdot(\nabla_{\mathbb{H}}\phi + \nabla_{\mathbb{H},y}\phi_1)\,dxdy \\ \qquad + \int_{\Omega\times Y} A(y)\nabla_{\mathbb{H},y}V_1\nabla_{\mathbb{H},y}w\,dxdy + \int_{\Omega\times Y}(v + V_1)(\phi + w)\,dxdy \\ \displaystyle = \int_{\Omega\times Y}(u + U_1 - u_d)(\phi + w)\,dxdy \end{cases} \tag{4.39}$$

*for all $(\phi, \phi_1, w) \in H^1_{\mathbb{H}}(\Omega) \times L^2(\Omega; H^1_{\#,\mathbb{H}}(M)/\mathbb{R}) \times L^2(\Omega; H^1_{0,\mathbb{H}}(B))$.*

The variational forms (4.39) are the optimality system corresponding to the following optimal control problem: find $(\bar{u},\bar{U}_1,\bar{\theta},\bar{\theta}_1)\in H^1_{\mathbb{H}}(\Omega)\times L^2(\Omega;H^1_{0,\mathbb{H}}(B))\times L^2(\Omega)\times L^2(\Omega\times B)$ such that $J(\bar{u},\bar{U}_1,\bar{\theta},\bar{\theta}_1)=\inf\{J(u,U_1,\theta,\theta_1)\}$, where

$$J(u,U_1,\theta,\theta_1)=\frac{1}{2}\int_{\Omega\times Y}(u+\chi_B U_1-u_d)^2\,dxdy+\frac{\rho}{2}\int_{\Omega\times B}|\theta+\theta_1|^2\,dxdy, \tag{4.40}$$

and $(u,U_1,\theta,\theta_1)$ satisfies the following variational form,

$$\begin{cases}\displaystyle\int_\Omega A_0\nabla_{\mathbb{H}}u\nabla_{\mathbb{H}}\phi\,dx+\int_{\Omega\times B}A(y)\nabla_{\mathbb{H},y}U_1\nabla_{\mathbb{H},y}w\,dxdy\\ \displaystyle\qquad+\int_{\Omega\times Y}(u+\chi_B U_1)(\phi+\chi_B w)\,dxdy\\ \displaystyle=\int_{\Omega\times Y}(f+\chi_B(\theta(x)+\theta_1))\,(\phi+\chi_B w)\,dxdy\end{cases} \tag{4.41}$$

for all $(\phi,w)\in H^1_{\mathbb{H}}(\Omega)\times L^2(\Omega;H^1_{0,\mathbb{H}}(B))$.

In contrast to the previous case, here the full scale separation is not possible. This is because, if we put $\phi=0$ in (4.41), the right-hand side reduces to $\int_B(f(x)+\chi_B(y)(\theta(x)+\theta_1(x,y))w(x,y)\,dxdy$. Like before, we cannot introduce a cell problem to write an explicit form of $U_1$, as $\theta_1$ is dependent on the microvariable $y$. That is why scales separation is looking difficult to us for the time being. The appearance of control with respect to the $y$ variable is something new in this scenario, perhaps related to control problems in the cell which needs further investigation.

*Remark:* Throughout this article, we have considered the standard Heisenberg group just to make the presentation simpler. This work can be carried out in any $n$-dimensional Heisenberg group following a similar path.

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